CLASSICAL DYNAMICS IN CURVED SPACE DYNAMICAL GROUPS, C.C.R. AND GEOMETRIC QUANTIZATION

By

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TABLE OF CONTENTS

ACKNOWLEDGMENTS .... [illegible]
ABSTRACT v

CHAPTERS

1 INTRODUCTION 1
2 SYMMETRY GROUPS ON RIEMANNIAN MANIFOLDS AND EXTENSIONS 5
3 SYMMETRY–VS–DYNAMICAL GROUPS OF TRANSFORMATIONS 11
4 CANONICAL LIE ALGEBRA ACTIONS AND PARALLELIZATIONS OF PHASE SPACE.. 14
5 THE DYNAMICAL GROUP G_c 19
6 THE WEYL–HEISENBERG GROUP W_c IN CURVED SPACE 29
7 HARMONIC ANALYSIS ON W_c AND GEOMETRIC QUANTIZATION 41
CONCLUSION 50
APPENDIX 52
REFERENCES 55
BIOGRAPHICAL SKETCH 57

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This research addresses the problem of quantization on a non euclidean environment using tools from symplectic mechanics, Lie group theory and Geometric Quantization. For the sake of simplicity and definiteness, we will consider the Poincaré half plane as model configuration space Q_c. The thrust of this dissertation concentrates on the possibility of performing a mathematically sound quantization in spaces of constant negative curvature by using a differential geometric approach to find a new set of C.C.R., the premise being that a general quantization on an arbitrary configuration space Q must include the determination of an appropriate analogue of the flat C.C.R.

We obtain a Lie group G_c which acts canonically, transitive and freely on the phase T^*Q_c, and includes the symmetries of the free dynamics hamiltonian. We expand on the cohomological reason for the necessity of central extensions of the dynamical group and compute the previously unknown extension W_c of the curved-space dynamical group G_c, which we call the Weyl-Heisenberg group of the dynamics.

We finally show that there is a generalization of the Stone-von Neumann theorem for the manifold Q_c, analogous to the one known for $Q_0 \cong \mathbb{R}^n$ in that there is a one-parameter family (parametrized by the value of $\hbar$) of inequivalent classes of irreducible unitary representations of the Weyl-Heisenberg group W_c. Each of these classes is equivalent to an irreducible representation obtained from a co-adjoint orbit $\mathcal{O} \subset \mathcal{W}_c^*$ by induction of a character on the orbit, chosen together with a strongly admissible real polarization, that always exist since we show that G_c is a solvable, simply connected, exponential Lie group

CHAPTER 1
INTRODUCTION

This research addresses the problem of quantization on a non-euclidean environment using tools from symplectic mechanics, Lie group theory and Geometric Quantization. One of the motivations for our research is that the correspondence between classical and Quantum Mechanics should be based on an analogy in their formal mathematical structures, and not just on a coincidence of their results when $\hbar \to 0$. Thus, we believe that any quantization should preserve and reflect the intrinsic geometric and dynamical symmetry properties of the classical description. Our first task will then be to focus on those symmetries of the classical setting that should be preserved in the quantum environment.

In Quantum Mechanics, a physical system with symmetries is called *elementary* if the symmetry group is *irreducibly* represented in the quantum phase space. This irreducibility condition should be viewed as the analogous to *transitivity*, at the classical level, of the group action on the classical phase space M (or some submanifold thereof specified by additional constraints on the dynamics). Moreover, within the symplectic formalization of hamiltonian dynamics already emerges a natural correspondence χ between the classical *observables* $f \in \mathcal{C}^{\infty}_{\mathbb{R}}(M)$ and first-order differential operators $\chi(f) = \mathbf{X}_f \in \mathcal{H}am(M)$, their *hamiltonian* vector fields relative to the symplectic structure ω attached to M (i.e., the solution to the exterior equation $\iota_{\mathbf{X}}\omega = -df$). Although χ is only a Lie algebra morphism — because any constant function in $\mathcal{C}^{\infty}_{\mathbb{R}}(M)$ has the zero field as image under χ (whereas for the purposes of quantization one requires a correspondence that is a monomorphism) — it is nevertheless true that it associates to each classical observable f a *1 parameter group of canonical transformations*: the flow of the hamiltonian vector field $\mathbf{X}_f$. The initial motivations for the traditional Dirac quantization of mechanics on a flat, Euclidean configuration space $Q_0 \cong \mathbb{R}^n$ were drawn from his recognition of similar analogies between the classical Poisson brackets of basic set of functions in $C^{\infty}_{\mathbb{R}}(T^*\mathbb{R}^n)$ and the commutator bracket of the corresponding quantum operators. In this case the dynamical group is also a (sub)group of symmetries of the dynamics and thus to each of its generators is attached a physical *observable*, the hamiltonian function associated with the corresponding invariant hamiltonian vector field on $T^*\mathbb{R}^n$. In particular, for $\mathbf{a}$ constant in $\mathbb{R}^n$, the linear functions $\mathbf{a}\cdot\mathbf{p}$ (mod$\mathbb{R}$) of the canonical momenta $\mathbf{p}$ (the Darboux coordinates in $T^*\mathbb{R}^n$ canonically conjugate to the *position* coordinates $\mathbf{q}$, and for which the symplectic structure is written as $\omega = d\mathbf{p}\wedge d\mathbf{q}$) are also the hamiltonian functions of the vector fields $\tilde{\mathbf{X}}_a = \mathbf{a}\cdot\partial_{\mathbf{q}} \in T(T^*\mathbb{R}^n)$ that correspond to generators of *translation isometries* $\mathbf{q} \to \mathbf{q}+\mathbf{a}$ in a configuration space $Q_0 \cong \mathbb{R}^n$. However, for a curved n-dimensional configuration space Q_c it is easily seen that no such correspondence exists in general and that the proper interpretation of the role of certain physical observables presents interesting problems. Indeed, the commutator (Lie bracket) of $\tilde{\mathbf{X}}_a = \mathbf{a}\cdot\partial_{\mathbf{q}}$ and $\tilde{\mathbf{X}}_b = \mathbf{b}\cdot\partial_{\mathbf{q}}$ (viewed as vector fields on T^*Q_c, $\mathbf{a}$, $\mathbf{b}$ constants in $\mathbb{R}^n$) is identically zero and therefore cannot be considered as *generators of translations* in Q_c since *curvature* prevents these from commuting.

In fact, it has been known for a long time that the usual Canonical Commutation Relations are only appropriate for systems which possess a classical configuration space admitting an *Abelian* transitive group of transformations. Since the recognition of the limitations of the Dirac quantization scheme by Groenwold (1946) [Gro] and van Hove (1951) [Hov], the resolution of the problems originated by the *no go* theorems that afflict a mathematically rigorous quantization has been the concern of the Geometric Quantization programme originated by Souriau (1966–1970) [Sou] and Kostant (1970) [Kos]. The work of those authors suggests that the study of symplectic manifolds that support a canonical and transitive action of a Lie group G, i.e., the G-homogeneous symplectic manifolds, can help clarify the traditional quantization procedures in physics which seek a direct correspondence between classical observables and self adjoint operators (the infinitesimal generators of *1-parameter groups of unitary transformations* on the quantum phase space)

In Chapter 2, we start our study of the free dynamics on non-flat Riemannian manifolds. For simplicity we will consider the Poincaré half-plane as model, which is *the* simply-connected homogeneous manifold of dimension 2 and constant negative curvature $k = -\frac{1}{c^2}$ The study of the dimensionality of the symmetry groups of the free dynamics in an n-dimensional Riemannian manifold M reveals that, in the case of manifolds whose sectional curvature is everywhere negative, there exists always a group of isometries acting transitively [Aze]. In particular manifolds of constant (scalar) curvature always allow the maximal number $\frac{1}{2}n(n+1)$ of independent Killing vector fields. Reciprocally an n-dimensional manifold with $\frac{1}{2}n(n+1)$ Killing vector fields must have constant (scalar) curvature The classical flat case is a particular example of this and we shall see in Chapter 3 how the translation subgroup $\mathbb{R}^n$ of the full space isometry group – the Euclidean group $E(\mathbb{R}^n)$ – leads, by a natural extension, to a larger group of symmetries $\mathbb{R}^{n}$, acting canonically, transitively and freely on phase space $T^*\mathbb{R}^n$, which is thus seen as a $\mathbb{R}^{2n}$–homogeneous manifold In Chapter 4 we propose a method to solve the questions raised previously by introducing the concept of *parallelization*, defined along the lines suggested by Marmo et al [Mar], we formulate mathematically the additional requirements to obtain Lie algebra parallelization of the cotangent bundle to a configuration space in which there is a Lie group of symmetries acting transitively, with the *proviso* that such a parallelization be formed by hamiltonian vector fields. In Chapter 5 we find an explicit solution for the problem formulated in the context of our model dynamical system, and we integrate the corresponding Maurer-Cartan equations to find the multiplication law of a Lie group whose Lie algebra is the specified Lie parallelization. We therefore obtain a Lie group G_c which acts canonically, transitive and freely on the phase T^*Q_c, which includes the symmetries of the free dynamics hamiltonian and which is shown to be a simply-connected, solvable and exponential Lie group.

Chapter 6 is devoted to the comparison of the algebraic structures involved in the geometric quantization process, with particular emphasis on the cohomological reason for the necessity of central extensions of the dynamical group, both in the flat and curved case; the result of this is the appearance of the familiar Weyl-Heisenberg group in the flat case setting, and we compute the previously unknown extension W_c of the curved-space dynamical group G_c. The central extension W_c will also be called the Weyl Heisenberg group of the dynamics.

One of the main reasons for the success of the traditional Quantum Mechanics lies with the Stone-von Neumann theorem, which guarantees that, after fixing the value of the Planck's constant $\hbar$, there is a unique equivalence class of weakly continuous, irreducible representations of the Weyl Heisenberg group. If that were not the case, Quantum Mechanics would need to be supplemented with a set of rules to specify how the correct irreducible representation class was to be selected, since inequivalant representations would lead to different physical predictions. In Chapter 7 we compute the coadjoint orbits of the group W_c in order to apply orbit-theory methods and obtain a classification of its irreducible representations. Due to the characteristics of W_c, we can finally prove a generalization of the Stone-von Neuman theorem to the kinematics on the manifold Q_c.

CHAPTER 2
SYMMETRY GROUPS
ON RIEMANNIAN MANIFOLDS AND EXTENSIONS

In the sequel we will consider mainly the case of configuration spaces of constant negative curvature. To ensure that the structures originated in this process arise in a natural way, we will guide our constructive and analytical procedures by modelling on a particular, amenable, non-euclidean setting: the free dynamics on a simply connected homogeneous Riemannian manifold of dimension 2 and constant negative curvature $-\frac{1}{c^2}$ i.e. the Poincaré half-plane. We choose the particular parametrization

[1] $$Q_c \equiv \left\{ \mathbf{q} \in \mathbb{R}^2 \;\; q^2 > -c;\; ds^2 = \left(\tfrac{c}{c+q^2}\right)^2 \delta_{ij}\, dq^i dq^j \right\}$$

Throughout this work we will find it useful to use the notation

[2] $$ds^2 = d\mathbf{q} \otimes d\mathbf{q}\;\; \mathbf{g}_c$$

with

[3] $$\mathbf{g}_c(\mathbf{q}) = \left(\tfrac{c}{c+q^2}\right)^2 \delta^* \qquad \text{and} \qquad \delta^* = \delta_{ij}\, \mathbf{e}^i \otimes \mathbf{e}^j \in \left(\mathbb{R}^2 \otimes \mathbb{R}^2\right)^*$$

The Lagrangean description of this system is given by the usual symplectic formalism on the tangent bundle $TQ_c \equiv Q_c \times \mathbb{R}^2$ of the configuration space Q_c. In the global chart $\{\mathbb{R}^2 \times \mathbb{R}^2, (\mathbf{q}, \mathbf{v})\}$ that naturally exists for the velocity space TQ_c, the free dynamics Lagrangean is

[4] $$\mathcal{L}(\mathbf{q}, \mathbf{v}) = \tfrac{1}{2}(\mathbf{v} \otimes \mathbf{v})\, \mathbf{g}_c(\mathbf{q})$$

To this Lagrangean is associated the symplectic form

[5] $$\omega_{\mathcal{L}} = -\, d\mathbf{q} \wedge d\mathbf{q}\, (\, \partial_q \partial_v\, \mathcal{L}\,) \quad d\mathbf{v} \wedge d\mathbf{q}\, (\, \partial_v \partial_v\, \mathcal{L}\,)$$

In coordinates this is

[6] $$\omega_{\mathcal{L}}\big|_{(\mathbf{q},\mathbf{v})} = \left(\frac{c}{c+q^2}\right)^2 \left[dq^1\wedge dv^1 + dq^2\wedge dv^2 - \frac{2v^1}{c+q^2}\, dq^1\wedge dq^2\right]$$

We can also consider the hamiltonian description, which has the advantage of allowing a formalism that does not depend *explicitly* on the particular choice of hamiltonian for the dynamics: the symplectic structure on phase space – the cotangent bundle T^*Q_c – is canonically determined, and there always exists a special (Darboux) coordinate system $(\mathbf{q},\mathbf{p})$ in which this structure is locally

[7] $$\omega = d\mathbf{p}\wedge d\mathbf{q}\ 1$$

where $1 = \mathbf{e}_1\otimes\mathbf{e}^1 + \mathbf{e}_2\otimes\mathbf{e}^2$. Of course, the 'physics' is included in the definition of the momenta $\mathbf{p}$ as a function of $\mathbf{q}$ and $\mathbf{v}$, and in the case at hand this is

[8] $$\mathbf{p}\,(q,v) = \partial_{\mathbf{v}}\mathcal{L} = \mathbf{v}\cdot g_c(\mathbf{q}) = \left(\frac{c}{c+q^2}\right)^2 \mathbf{v}\cdot\delta^*$$

We can thus write the hamiltonian function $\mathcal{H}\in C^{\infty}_{\mathbb{R}}(T^*Q_c)$ as

[9] $$\mathcal{H}(\mathbf{q},\mathbf{p}) = \tfrac{1}{2}(\mathbf{p}\otimes\mathbf{p})\, g_c^*(\mathbf{q})$$

where

[10] $$g_c^*(\mathbf{q}) = \left(\frac{c+q^2}{c}\right)^2\delta \quad\text{and}\quad \delta = \delta^{ij}\,\mathbf{e}_i\otimes\mathbf{e}_j \in \mathbb{R}^2\otimes\mathbb{R}^2$$

The corresponding hamiltonian vector field relative to the canonical symplectic structure is

[11] $$\mathbf{X}_{\mathcal{H}} = (\partial_{\mathbf{p}}\mathcal{H})\,\partial_{\mathbf{q}} - (\partial_{\mathbf{q}}\mathcal{H})\,\partial_{\mathbf{p}} = \left(\frac{c+q^2}{c}\right)^2 \mathbf{p}\cdot\partial_{\mathbf{q}} - \left(\frac{c+q^2}{c}\right)(\mathbf{p}\otimes\mathbf{p})\,\delta\,\partial_{p_2}$$

Given the metric structure of configuration space Q_c, we first look at its isometries and determine the set of Killing vector fields $\mathbf{X}\in TQ_c$, i.e. the solutions to the equations

[12] $$\mathfrak{L}_{\mathbf{X}}\,(ds^2) = 0$$

where $\mathfrak{L}$ denotes the Lie derivative along $\mathbf{X}$

Explicitly for

$$[13] \qquad \mathbf{X} = \mathbf{x}\cdot\partial_{\mathbf{q}} \in TQ_c \qquad \text{and} \qquad \mathbf{x} = X^i(\mathbf{q})\,\mathbf{e}_i \in \mathcal{F}(Q_c, \mathbb{R}^2)$$

equation [12] reads

$$[14] \qquad X^i \partial_i\, g_{rs} + g_{is}\,\partial_r X^i + g_{ri}\,\partial_s X^i = 0$$

For the case at hand this reduces further to

$$[15] \qquad \begin{cases} \partial_1 X^2 = -\,\partial_2 X^1 & \text{for } (r{\neq}s) \\ \partial_1 X^1 = \;\partial_2 X^2 = \left(\frac{1}{c+q^2}\right) X^2 & \text{for } (r{=}s) \end{cases}$$

The general solution is

$$[16] \qquad \begin{cases} X^1(\mathbf{q}) = \frac{1}{2c}\left\{(q^1)^2 - (q^2+c)^2\right\}\eta_1 + \frac{1}{c}q^1\eta_2 + \eta_3 \\ X^2(\mathbf{q}) = \frac{1}{c}(c+q^2)(q^1\eta_1 + \eta_2) \end{cases}$$

where η_1, η_2, η_3 are arbitrary constants. We will denote by $\mathbf{X}_{\boldsymbol{\eta}}$ a vector in $\mathcal{K}il(Q_c)$ defined by a particular $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$. The Killing vectors thus obtained form a 3-dimensional Lie algebra $\mathcal{K}il(Q_c)$ which is isomorphic to $sl(2,\mathbb{R})$. In fact, we can readily find a basis $\{\mathbf{X}_{\boldsymbol{\eta}^+}, \mathbf{X}_{\boldsymbol{\eta}^-}, \mathbf{X}_{\boldsymbol{\eta}^0}\}$ of $\mathcal{K}il(Q_c)$ verifying the $sl(2,\mathbb{R})$-commutation relations

$$[17] \qquad [\mathbf{X}_{\eta^+}, \mathbf{X}_{\eta^-}] = \mathbf{X}_{\eta^0}\,, \qquad [\mathbf{X}_{\eta^0}, \mathbf{X}_{\eta^+}] = \mathbf{X}_{\eta^+} \qquad [\mathbf{X}_{\eta^0}, \mathbf{X}_{\eta^-}] = -\mathbf{X}_{\eta^-}$$

However, we will be more interested in a basis of $\mathcal{K}il(Q_c)$ that, in the flat case limit $(c \to \infty)$, deforms into $\mathcal{E}(\mathbb{R}^2)$: the algebra of the Euclidean group in two dimensions. This will then be a basis of generators of isometries that can readily be interpreted as the analog of the natural basis of $\mathcal{E}(\mathbb{R}^2)$, i.e. the generators of space translations and rotations

In fact any basis $\{\mathbf{X}_{\eta_1},\mathbf{X}_{\eta_2},\mathbf{X}_{\eta_3}\}$ is determined up to a choice of $\alpha, \alpha, \beta, \gamma \in \mathbb{R}$ by

$$[18] \qquad \begin{cases} \eta_1 = (\frac{1}{c}\alpha, \frac{1}{c}\alpha, 1+\frac{1}{2}\alpha) & \text{for} \quad \mathbf{X}_{\eta_1} \\ \eta_2 = (\frac{1}{c}\beta, 1, \frac{1}{2}\beta) & \text{for} \quad \mathbf{X}_{\eta_2} \\ \eta_3 = (1, \frac{1}{c}\gamma, \frac{c}{2}) & \text{for} \quad \mathbf{X}_{\eta_3} \end{cases}$$

These basis vectors go over the apropriate generators of $\mathcal{E}(\mathbb{R}^2)$ in the $(c\to\infty)$ limit

$$[19] \qquad \begin{cases} \mathbf{X}_{\eta_1} \to \mathbf{X}_{p_1}|_{(q)} = \partial_{q^1} \\ \mathbf{X}_{\eta_2} \to \mathbf{X}_{p_2}|_{(q)} = \partial_{q^2} \\ \mathbf{X}_{\eta_3} \to \mathbf{X}_{L}|_{(q)} = q^1\partial_{q^2} - q^2\partial_{q^1} \end{cases}$$

Not every set [18] generates a *Lie algebra* basis $\{\mathbf{X}_{\eta_1},\mathbf{X}_{\eta_2},\mathbf{X}_{\eta_3}\}$ in general, but if we choose the set $\eta_1 = (0,0,1)$, $\eta_2 = (0,1,0)$, $\eta_3 = (1,0,\frac{c}{2})$ they will, in which case we will denote the corresponding basis vectors as $\{\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3\}$ where

$$[20] \qquad \begin{cases} \mathbf{X}_1|_{(q)} = \partial_{q^1} \\ \mathbf{X}_2|_{(q)} = \frac{1}{c}q^1\partial_{q^1} + (1+\frac{1}{c}q^2)\partial_{q^2} \\ \mathbf{X}_3|_{(q)} = \frac{1}{2c}\{(q^1)^2-(q^2)^2-2cq^2\}\partial_{q^1} + q^1(1+\frac{1}{c}q^2)\partial_{q^2} \end{cases}$$

This basis verifies the commutation relations

$$[21] \qquad [\mathbf{X}_1,\mathbf{X}_2] = \tfrac{1}{c}\mathbf{X}_1 \quad [\mathbf{X}_1,\mathbf{X}_3] = \mathbf{X}_2\ , \quad [\mathbf{X}_2,\mathbf{X}_3] = -\mathbf{X}_1 + \tfrac{1}{c}\mathbf{X}_3$$

As expected, in the limit $(c\to\infty)$ these become the defining relations for the generators of the Euclidean group $\mathcal{E}(\mathbb{R}^2)$. However, these commutation relations do determine, for $c<\infty$ a Lie algebra homeomorphic to $sl(2,\mathbb{R})$. The usual basis can be obtained as

$$[22] \qquad \mathbf{X}_+ = -c\,\mathbf{X}_1 \qquad \mathbf{X}_- = \mathbf{X}_3 - \tfrac{1}{c}\mathbf{X}_1\ ; \qquad \mathbf{H} = -c\,\mathbf{X}_2$$

which verify the standard commutation relations

$$[23] \qquad [\mathbf{X}_+,\mathbf{X}_-] = \mathbf{H} \quad [\mathbf{H},\mathbf{X}_+] = \mathbf{X}_+ \quad [\mathbf{H},\mathbf{X}_-] = -\mathbf{X}_-$$

From the commutation relations [21], we verify that $\{\mathbf{X}_1,\mathbf{X}_2\}$ generates a solvable sub-algebra $\mathcal{H}_c$ of $\mathcal{K}il(Q_c)$, and the restriction to this sub-algebra of the Lie algebra action of $\mathcal{K}il(Q_c)$ on Q_c can be integrated to yield a free, transitive Lie-group action of the simply connected Lie group $H_c = \exp(\mathcal{H}_c)$ on Q_c. In canonical coordinates

[24] $$\mathbf{h} = \exp(t\,\mathbf{X}_a)\,|_{(t=1)} \in \mathcal{F}(\mathbf{H}_c, \mathbb{R}^2)$$

where $\mathbf{X}_a = \sum_i a^i\,\mathbf{X}_i \in \mathcal{H}_c$ and

[25] $$\exp(t\,\mathbf{X}_a) = \tfrac{c}{a^2}\left(e^{\frac{ta^2}{c}} - 1\right)\mathbf{a}$$

denotes the flow of $\mathbf{X}_a \in \mathcal{H}_c$, with $\mathbf{a} = (a^1, a^2) \in \mathbb{R}^2$, $t \in \mathbb{R}$. The group $\mathbf{H}_c$ is defined by the set $H_c \equiv \{\mathbf{h} \in \mathbb{R}^2 \;\; h^2 > -c\}$ with multiplication law

[26] $$\mathbf{h}\mathbf{h}' = \left(1 + \tfrac{h^2}{c}\right)\mathbf{h}' + \mathbf{h}$$

This group acts isometrically on Q_c through the map

[27] $$\begin{aligned} \Phi \;\; & H_c \times Q_c \longrightarrow Q_c \\ (\mathbf{h}\,,\mathbf{q}\,) \;\; & \mapsto \Phi_{\mathbf{h}}(\mathbf{q}) = \left(1 + \tfrac{h^2}{c}\right)\mathbf{q} + \mathbf{h} \end{aligned}$$

Thus, we can identify the group manifold $\mathbf{H}_c$ with the configuration space manifold Q_c. This also happens trivially in the flat case, where the additive translation group $\mathbb{R}^2$ plays the role of H_c; in fact, $\mathbf{H}_c$ deforms — both as a group and Riemannian manifold — to its Cartesian counterpart $\mathbb{R}^2$.

We now analyze the way symmetries in configuration space generate dynamical symmetries on phase space (T^*Q_c) or velocity space (TQ_c). We will see later on that, already at the level of classical mechanics in $\mathbb{R}^n$, we can trace the appearance of the Heisenberg group to the existence of a *transitive canonical action* of the sub-group $\mathbb{R}^{2n}$ of the Galilei group, on the phase-space of the free dynamics in $\mathbb{R}^n$. It is then an interesting problem to determine the possible equivalents of these objects in the case of a free dynamics on a space that is no longer flat. In particular, we aim to determine in this manner the essential features of $\mathbb{R}^{2n}$ that should be preserved in order to obtain a transitive canonical action of a dynamical group on phase space when we consider the case where the configuration space is a homogeneous Riemann manifold.

Evidently, since we have seen that we can view a particular normal subgroup of the full isometry group to be the analog of the sub-group of space translations in $\mathbb{R}^{2n}$, the first impulse is to find an equivalent for *boosts* on a dynamical group-analog for the Galilei group. There are however no natural generalizations of the Galilean principle of relativity to the non-flat environment [Cha]. We therefore investigate in Chapter 2 whether, even in the flat case, there is a way to construct such a dynamical group, based solely on information from the isometries of configuration space.

CHAPTER 3
SYMMETRY vs DYNAMICAL GROUPS OF TRANSFORMATIONS

Configuration space in ordinary classical mechanics is a flat, Euclidean space isometric to $\mathbb{R}^n$. Time is an independent quantity introducing an extra coordinate, and the time dependent dynamics is describable in terms of a contact structure on $(T^*\mathbb{R}^n)\times\mathbb{R}$. The natural contact 2-form can be generally written $\tilde{\omega} = \pi^*\omega - d\mathcal{H}\wedge dt$, where $\omega = d\mathbf{p}\wedge d\mathbf{q}$, and $\pi : (T^*\mathbb{R}^n)\times\mathbb{R} \longrightarrow T^*\mathbb{R}^n$. More generally, a classical hamiltonian description of mechanics on a configuration manifold $\mathbf{Q}\times\mathbb{R}$ would be based on the corresponding cotangent $T^*(\mathbf{Q}\times\mathbb{R})$ bundle. Since this is a trivial bundle over $(T^*\mathbf{Q})\times\mathbb{R}$, a dynamical system characterized by an hamiltonian $\mathcal{H}(\mathbf{q},\mathbf{p},t)$ determines a cross-section

[28] $$\begin{aligned} \sigma : (T^*\mathbf{Q})\times\mathbb{R} &\longrightarrow T^*(\mathbf{Q}\times\mathbb{R}) \\ (\mathbf{q},\mathbf{p},t) &\mapsto \sigma(\mathbf{q},\mathbf{p},t) = (\mathbf{q},\mathbf{p},t,-\mathcal{H}(\mathbf{q},\mathbf{p},t)) \end{aligned}$$

If we denote by Ω the natural symplectic form on $T^*(\mathbf{Q}\times\mathbb{R})$, its pull-back by σ is precisely

[29] $$\sigma^*\Omega\big|_{(\mathbf{q},\mathbf{p},t)} = d\mathbf{p}\wedge d\mathbf{q} - d\mathcal{H}(\mathbf{q},\mathbf{p},t)\wedge dt = \tilde{\omega}\big|_{(\mathbf{q},\mathbf{p},t)}$$

The evolution of the system is then determined by the flow of the dynamical vector field $\tilde{\mathbf{X}}_{\mathcal{H}}$ tangent to $(T^*\mathbf{Q})\times\mathbb{R}$, associated with the hamiltonian $\mathcal{H}$ by

[30] $$\iota_{\tilde{\mathbf{X}}_{\mathcal{H}}}\tilde{\omega} = 0$$

where ι denotes the inner product (or contraction) between $\tilde{\mathbf{X}}_{\mathcal{H}}$ and $\tilde{\omega}$.

We then consider a time-dependent formulation of dynamics in the flat case, where the evolution of the state of a system is given by the flow of the hamiltonian vector field

$\tilde{\mathbf{X}}_{\mathcal{H}} \in T(T^{*}\mathbb{R}^{n} \times \mathbb{R})$ which annihilates the contact structure on $(T^{*}Q)\times\mathbb{R}$ (the evolution space)

[31] $$\tilde{\omega} = d\mathbf{p}\wedge d\mathbf{q}\,1 - d\mathcal{H}\wedge dt$$

where $\mathcal{H}$ is the hamiltonian for the system. For free systems,

[32] $$\tilde{\omega} = d\mathbf{p}\wedge d\mathbf{q}\,1 - \tfrac{1}{2}\mathbf{p}\cdot d\mathbf{p}\wedge dt$$

We look for symmetry generators $\tilde{\mathbf{X}} \in T(T^{*}\mathbb{R}^{n}\times\mathbb{R})$ such that

i) $\mathcal{L}_{\tilde{\mathbf{X}}}\tilde{\omega} = 0$

ii) these vectors generate transformations that, when projected to configuration space, are isometries of the Euclidean metric on $\mathbb{R}^{n}$

iii) when projected to phase space, these vectors generate canonical transformations of the symplectic structure $\omega = d\mathbf{p}\wedge d\mathbf{q}\,1$

If we write $\tilde{\mathbf{X}}\,|_{(\mathbf{q},\mathbf{p},t)} = \mathbf{u}(\mathbf{q},\mathbf{p},t)\cdot\partial_{\mathbf{q}} + \mathbf{a}(\mathbf{q},\mathbf{p},t)\cdot\partial_{\mathbf{p}} + \tau(\mathbf{q},\mathbf{p},t)\,\partial_{t}$, the last two conditions restrict $\mathbf{u}$ and $\mathbf{a}$ to be of the form

[33] $$\mathbf{u}(\mathbf{q},t) = A(t)\cdot\mathbf{q} + \mathbf{d}(t) \qquad A(t) \in SO(n)$$

[34] $$\mathbf{a}(\mathbf{q},\mathbf{p},t) = -(\mathbf{p}\cdot\partial_{\mathbf{q}} + \partial_{t})\mathbf{u}(\mathbf{q},t) \qquad \mathbf{d}(t) \in \mathbb{R}^{n}$$

Solving the set of P.D.E.'s defined by i) we obtain, as general solution, that

[35] $$A'(t) = 0 \quad , \quad \mathbf{d}''(t) = 0 \quad , \quad \tau(\mathbf{q},\mathbf{p},t) = \nu + \tfrac{1}{2}\mathbf{p}^{2}\lambda = \tau$$

We therefore see that any solution $\tilde{\mathbf{X}}$ verifying the conditions i)-iii) above must be a linear combination of

i) generators of space translations $\mathbf{d}_{0}\cdot\partial_{\mathbf{q}}$,

ii) generators of momentum translations (*boosts*) $\mathbf{b}_{0}\cdot(t\,\partial_{\mathbf{q}} + \partial_{\mathbf{p}})$,

iii) generators of rotations $(A_{0}\cdot\mathbf{q})\cdot\partial_{\mathbf{q}} + (A_{0}\cdot\mathbf{p})\,\partial_{\mathbf{p}}$

iv) generators of time translations $\tau_{0}\cdot\partial_{t}$

where $\mathbf{d}_0, \mathbf{b}_0 \in \mathbb{R}^n$, $A_0 \in SO(n)$, $\tau \in \mathbb{R}$ are constants. In the 2-dimensional flat analogue, the 3-dimensional Euclidean group $E_2 = SO(2) \circledS \mathbb{R}^2$ thus extends to a 6-dimensional symmetry group $G = E_2 \times (\mathbb{R}^2 \times \mathbb{R})$ with a group law

[36] $$(A_0, \mathbf{d}_0, \mathbf{b}_0, \tau_0)(A, \mathbf{d}, \mathbf{b}, \tau) = (A_0 A, A\mathbf{d}_0 + \mathbf{d} + \mathbf{b}\tau_0, A\mathbf{b}_0 + \mathbf{b}, \tau + \tau_0)$$

in which the boost translation subgroup $\mathbb{R}^2 \times \mathbb{R}^2$ is included as an abelian subgroup acting transitively and freely in phase space. We therefore obtain the Galilei group explicitly solely from the use of space isometries and the invariance of the dynamics.

This suggests that we attempt to solve the same problem for the non flat case Q_c, substituting the Euclidean metric by the hyperbolic metric $ds^2 = \mathbf{dq} \otimes \mathbf{dq}\, \mathbf{g}_c$ and setting $\mathcal{H}(\mathbf{q}, \mathbf{p}) = \frac{1}{2}(\mathbf{p} \otimes \mathbf{p})\, \mathbf{g}_c(\mathbf{q})$ on the contact form $\widetilde{\omega}$. Unfortunately this does not work. Lifting $sl(2, \mathbb{R})$ this way does not generate any more symmetries. We end up with the trivial lifting of $ksl(Q_c)$ to T^*Q_c, thus confirming from a different perspective that the concept of *boost* does not generalize [Cha] in this way to the curved-space dynamics. Thus we cannot find in a non-flat environment a dynamical group that is large enough and is entirely based on the symmetries of the configuration space. We therefore look for another structure that exists also in the non-flat case, but is hidden in $\mathbb{R}^{2n}$, and reflects the symmetry of the dynamics.

We propose that such a structure is provided by a Lie algebra *parallelization* of phase space, in fact one that naturally exists on the tangent or co-tangent space of any Lie group manifold G: this is the parallelization generated by either left- or right-invariant vector fields on G, discussed in Chapter 2.

CHAPTER 4
CANONICAL LIE ALGEBRA ACTIONS AND PARALLELIZATIONS OF PHASE SPACE

A manifold M is said to be parallelizable if a set of n=dim M independent vector fields can be globally defined on M. The 2-dimensional sphere S^2 is an example of a non-parallelizable manifold, whereas the 2-torus T^2 is parallelized by the set $\{\partial_\theta, \partial_\varphi\}$. Also, any Lie group is parallelizable, since the corresponding Lie algebra can always be realized as the set of left- or right-invariant vector fields on the group manifold. In general since the $\mathbf{X}_i$ are independent, for a given parallelization $\{\mathbf{X}_i\}_{(i=1...n)}$ of M we have

$$[37] \qquad [\mathbf{X}_i, \mathbf{X}_j] = c_{ij}{}^k \, \mathbf{X}_k$$

where $c_{ij}{}^k = -c_{ji}{}^k \in \mathcal{F}(M)$. However, unless the $\{c_{ij}{}^k\}$ are constants, the parallelization does not necessarily form a Lie algebra. Our aim in the next two sections is to obtain a parallelization of T^*H_c by 4 vector fields such that

a) among them there are two that satisfy the same commutation relations as the starting parallelization of H_c discussed in Chapter 2

b) the vector fields in this parallelization are hamiltonian with respect to the canonical symplectic structure on $T^*\mathbf{H}_c$

c) this parallelization forms a Lie algebra $\mathcal{G}_c$ whose action on T^*H_c is integrable to a canonical, transitive action of a Lie group G_c on T^*H_c

We will then be able to interpret the transitive group G_c acting canonically on T^*H_c as the integral of the Lie-algebra parallelization $\{\mathbf{X}_i\}_{(i=1...n)}$ just as, in the flat case, $\mathbb{R}^{2n}$ can be viewed as the group generated by the parallelization $\{\partial_{\mathbf{q}}, \partial_{\mathbf{p}}\}$. In the present Chapter we outline this program, and implement it explicitely in Chapter 5

If a manifold is parallelizable, it is always possible to parallelize its tangent or cotangent spaces in such a way that condition (a) is always verified. In fact, the canonical lifts of the vector fields that parallelize the base manifold obey the same commutation relations as their projections. Explicitly, denoting a parallelization of a manifold Q by $\wp(Q) = \{\mathbf{X}_a = \mathbf{u}_a\, \partial_{\mathbf{q}},\ \mathbf{u}_a \in \mathcal{F}(Q, \mathbb{R}^n),\ a=1,\dots,n=\dim(Q)\}$, we construct the lifted vector fields

$$[38]\qquad \mathbf{X}_a = \mathbf{u}_a\, \partial_{\mathbf{q}} + (\mathbf{v}\, \partial_{\mathbf{q}} \mathbf{u}_a)\, \partial_{\mathbf{v}} \in T(TQ)$$

$$[39]\qquad \tilde{\mathbf{X}}_a = \mathbf{u}_a\, \partial_{\mathbf{q}} - \{\partial_{\mathbf{q}}(\mathbf{u}_a\, \mathbf{p})\}\, \partial_{\mathbf{p}} \in T(T^*Q)$$

A simple calculation shows that each set in [38] or [39] has indeed the same commutation relations as the vectors $\mathbf{X}_a$ of the parallelization $\wp(Q)$. In $T(T^*Q)$ for example

$$[\tilde{\mathbf{X}}_a, \tilde{\mathbf{X}}_b] = \{\tilde{\mathbf{X}}_a(\mathbf{u}_b) - \tilde{\mathbf{X}}_b(\mathbf{u}_a)\}\, \partial_{\mathbf{q}} + \{\tilde{\mathbf{X}}_b(\partial_{\mathbf{q}}(\mathbf{u}_a\, \mathbf{p})) - \tilde{\mathbf{X}}_a(\partial_{\mathbf{q}}(\mathbf{u}_b\, \mathbf{p}))\}\, \partial_{\mathbf{p}}$$

$$= \{\mathbf{u}_a\, \partial_{\mathbf{q}} \mathbf{u}_b - \mathbf{u}_b\, \partial_{\mathbf{q}} \mathbf{u}_a\}\, \partial_{\mathbf{q}} + \{\partial_{\mathbf{q}}(\mathbf{u}_b\, \partial_{\mathbf{q}} \mathbf{u}_a - \mathbf{u}_a\, \partial_{\mathbf{q}} \mathbf{u}_b)\, \mathbf{p}\}\, \partial_{\mathbf{p}} =$$

$$= c_{ab}{}^{k}\, (\mathbf{u}_k\, \partial_{\mathbf{q}} - \{\partial_{\mathbf{q}}(\mathbf{u}_k\, \mathbf{p})\}\, \partial_{\mathbf{p}}) = c_{ab}{}^{k}\, \tilde{\mathbf{X}}_k$$

Furthermore these lifted vector fields are hamiltonian with respect to the canonical symplectic structure $\omega = d\mathbf{p} \wedge d\mathbf{q}$ on T^*Q. In fact

$$[40]\qquad \iota_{\tilde{\mathbf{X}}_a} \omega = -\, d\mathbf{p}\, \mathbf{u}_a - d\mathbf{q}\, \partial_{\mathbf{q}}(\mathbf{u}_a\, \mathbf{p}) = -\, d(\mathbf{u}_a\, \mathbf{p})$$

We will denote the hamiltonian function corresponding to $\tilde{\mathbf{X}}_a$, defined by eq. [40] up to an additive constant, by

$$[41]\qquad P_a(\mathbf{p},\mathbf{q}) = \mathbf{u}_a(\mathbf{q})\, \mathbf{p} + d_a$$

We then specify a choice of the remaining n vector fields for the case of T^*Q that has the advantage of deforming to the coordinate parallelization $\{\partial_{\mathbf{q}}, \partial_{\mathbf{p}}\}$ of $T^*\mathbb{R}^2$ in the flat case limit $(c \rightarrow \infty)$. This is

[42] $$\partial_{\mathbf{p}_a} = \hat{\mathbf{u}}^a\,\partial_{\mathbf{p}} \qquad \text{(for } a=1,\ldots,n\text{)}$$

where $\hat{\mathbf{u}}^a$ is dual to $\mathbf{u}_a$, i.e. $\hat{\mathbf{u}}^a\,\mathbf{u}_b = \delta^a_b$. Furthermore, this solution forms an abelian Lie algebra by itself, and the parallelization $\wp(T^*Q) \equiv \{\check{\mathbf{X}}_a, \partial_{\mathbf{p}_a}, a=1,\ldots, n=\dim(Q)\}$ has the commutation relations

[43] $$\begin{cases} [\check{\mathbf{X}}_a, \check{\mathbf{X}}_b] = c_{ab}{}^k\,\check{\mathbf{X}}_k \\ [\check{\mathbf{X}}_a, \partial_{\mathbf{p}_b}] = -\,c_{ak}{}^b\,\partial_{\mathbf{p}_k} \\ [\partial_{\mathbf{p}_a}, \partial_{\mathbf{p}_b}] = 0 \end{cases}$$

It only remains to show the second of the three relations [43]; this obtains as follows.

[44] $$[\check{\mathbf{X}}_a, \partial_{\mathbf{p}_b}] = \{\mathbf{u}_a\,\partial_{\mathbf{q}}\hat{\mathbf{u}}^b + (\partial_{\mathbf{q}}\mathbf{u}_a)\,\hat{\mathbf{u}}^b\}\,\partial_{\mathbf{p}} =$$
$$= \{(\mathbf{u}_k\,\partial_{\mathbf{q}}\mathbf{u}_a - \mathbf{u}_a\,\partial_{\mathbf{q}}\mathbf{u}_k)\,\hat{\mathbf{u}}^b \otimes \hat{\mathbf{u}}^k\}\,\partial_{\mathbf{p}} = -\,c_{ak}{}^b\,\hat{\mathbf{u}}^k\,\partial_{\mathbf{p}} =$$
$$= -\,c_{ak}{}^b\,\partial_{\mathbf{p}_k}$$

In this, we made use of the duality between $\mathbf{u}_a$ and $\hat{\mathbf{u}}^a$ to derive the identities

[45] $$\mathbf{u}_a \otimes \hat{\mathbf{u}}^a = 1 \quad , \qquad \partial_{\mathbf{q}}\hat{\mathbf{u}}^b = -\,(\partial_{\mathbf{q}}\mathbf{u}_a)\,\hat{\mathbf{u}}^b \otimes \hat{\mathbf{u}}^a\,.$$

Hence, if the vectors in the original parallelization $\wp(Q)$ form a Lie algebra with structure constants $\{c_{ab}{}^k\}$, then $\wp(T^*Q) \equiv \{\check{\mathbf{X}}_a, \check{\mathbf{X}}_{a+n} = \partial_{\mathbf{p}_a}, a=1,\ldots, n=\dim(Q)\}$ also defines a Lie algebra with structure constants $\{\check{c}_{st}{}^l\}$ given by

[46] $$\check{c}_{ab}{}^k = c_{ab}{}^k,\; \check{c}_{a\,b+n}{}^k = 0;\quad \check{c}_{a\,b+n}{}^{k+n} = -\,c_{ak}{}^b,\; \check{c}_{a+n\,b+n}{}^{k+n} = 0$$

for $a, b, k \in \{1,\ldots, n\}$

Unfortunately, in general the $\partial_{\mathbf{p}_a}$ are not hamiltonian vector fields for the canonical symplectic form $\omega = d\mathbf{p} \wedge d\mathbf{q}$; since the one form

[47] $$\iota_{\partial_{\mathbf{p}_a}}\omega = d\mathbf{q}\,\hat{\mathbf{u}}^a$$

is not necessarily closed. This is inconvenient, since then the vector fields $\partial_{\mathbf{p}_a}$ cannot be viewed as generators of canonical transformations, and thus the possibility of constructing a dynamical group from the Lie algebra parallelization [43] is compromised. We therefore look for vector fields

[48] $$\tilde{\mathbf{X}}_{a+n} = A_{(a+n)k}(\mathbf{q})\partial_{\mathbf{p}_k}$$

that are hamiltonian with respect to the canonical symplectic structure. The conditions for that to happen are derived from the closure of the one form

[49] $$\iota_{\tilde{\mathbf{X}}_{a+n}}\omega = d\mathbf{q}\,\{A_{(a+n)k}\hat{\mathbf{u}}^k\}$$

where

[50] $$\omega = d\mathbf{p}\wedge d\mathbf{q}\,.$$

Explicitely, this is

[51] $$d\mathbf{q}\wedge d\mathbf{q}\,\partial_q\{A_{(a+n)k}\hat{\mathbf{u}}^k\} = 0$$

which is equivalent to

[52] $$\partial_{q^i}\{A_{(a+n)k}\hat{u}^k_j\} - \partial_{q^j}\{A_{(a+n)k}\hat{u}^k_i\} = 0$$

or

[53] $$\left(\partial_{q^i}A_{(a+n)k}\right)\hat{u}^k_j - \left(\partial_{q^j}A_{(a+n)k}\right)\hat{u}^k_i = -A_{(a+n)k}\left(\partial_{q^i}\hat{u}^k_j - \partial_{q^j}\hat{u}^k_i\right)$$

Multiplication of both sides by $u_b{}^i u_c{}^j$ yields

[54] $$\begin{aligned} \mathbf{u}_b\left(\partial_{\mathbf{q}}A_{(a+n)c}\right) - \mathbf{u}_c\left(\partial_{\mathbf{q}}A_{(a+n)b}\right) &= \\ &= -A_{(a+n)k}\left\{\mathbf{u}_b\left(\partial_{\mathbf{q}}\hat{\mathbf{u}}^k\right)\mathbf{u}_c - \mathbf{u}_c\left(\partial_{\mathbf{q}}\hat{\mathbf{u}}^k\right)\mathbf{u}_b\right\} = \\ &= A_{(a+n)k}\left\{\mathbf{u}_b\left(\partial_{\mathbf{q}}\mathbf{u}_c\right) - \mathbf{u}_c\left(\partial_{\mathbf{q}}\mathbf{u}_b\right)\right\}\hat{\mathbf{u}}^k = \\ &= A_{(a+n)k}\,c_{bc}{}^d\,\delta^k_d \end{aligned}$$

[55] $$\mathbf{u}_b\left(\partial_{\mathbf{q}} A_{(a+n)e}\right) - \mathbf{u}_e\left(\partial_{\mathbf{q}} A_{(a+n)b}\right) = c_{be}{}^{d} A_{(a+n)d}$$

On the other hand, the commutation relations between vector fields $\breve{\mathbf{X}}_b$ and $\breve{\mathbf{X}}_{a+n}$ are, on account of [46] and [48]

[56] $$[\breve{\mathbf{X}}_b, \breve{\mathbf{X}}_{a+n}] = A_{(a+n)d}\,[\breve{\mathbf{X}}_b, \partial_{\mathbf{P}_d}] + \breve{\mathbf{X}}_b\left(A_{(a+n)e}\right)\partial_{\mathbf{P}_e} =$$

$$= \left(-\,c_{be}{}^{d} A_{(a+n)d} + \mathbf{u}_b\,\partial_{\mathbf{q}} A_{(a+n)e}\right)\partial_{\mathbf{P}_a} =$$

$$= \left(\mathbf{u}_e\,\partial_{\mathbf{q}} A_{(a+n)b}\right)\partial_{\mathbf{P}_e}$$

The conditions for having the $\breve{\mathbf{X}}_a$ and $\breve{\mathbf{X}}_{a+n}$, $(a=1,\dots,n)$ generating a Lie algebra of Hamiltonian vector fields are thus:

[57] $$\mathbf{u}_e\,\partial_{\mathbf{q}} A_{(a+n)b} = \breve{c}_{b\ a+n}{}^{d+n} A_{(d+n)e}$$

[58] $$\breve{c}_{e\ a+n}{}^{d+n} A_{(d+n)b} - \breve{c}_{b\ a+n}{}^{d+n} A_{(d+n)e} = c_{be}{}^{d} A_{(a+n)d}$$

where the $\breve{c}_{b\ a+n}{}^{d+n}$ are constants obeying the requirements of the structure constants of a Lie algebra

[59] $$\breve{c}_{b\ a+n}{}^{d+n} = -\,\breve{c}_{a+n\ b}{}^{d+n}$$

[60] $$\breve{c}_{b\ e+n}{}^{k+n}\,\breve{c}_{a\ k+n}{}^{d+n} + \breve{c}_{e+n\ a}{}^{k+n}\,\breve{c}_{b\ k+n}{}^{d+n} + \breve{c}_{a\ b}{}^{k}\,\breve{c}_{e+n\ k}{}^{d+n} = 0$$

In the next section, this program is carried out explicitely for the case of T^*Q_c

CHAPTER 5
THE DYNAMICAL GROUP G_c

Consider the Poincaré half plane Q_c on which there is a transitive Lie algebra action of the solvable sub-algebra $\mathcal{H}_c$ of $\mathcal{X}\ell(Q_c)$ generated by the vector fields $\{\mathbf{X}_1,\mathbf{X}_2\}$ defined in [20]. As previously indicated, this is a Lie-algebra parallelization that *is* integrable to a Lie group action, yielding a free, transitive, isometric action of the simply connected Lie group $\mathbf{H}_c = \exp(\mathcal{H}_c)$ on Q_c. We will henceforth denote this action on the *left* by

[61] $$\mathrm{L}_{\mathbf{h}}\,\mathbf{q} = \mathbf{h}\,\mathbf{q} = \left(1+\tfrac{\mathbf{h}^2}{c}\right)\mathbf{q} + \mathbf{h}$$

where $\mathbf{h} \in \mathrm{H}_c$ and $\mathbf{q} \in Q_c$. Since the group multiplication law on H_c is

[62] $$\mathbf{h}\,\mathbf{h}' = \left(1+\tfrac{\mathbf{h}^2}{c}\right)\mathbf{h}' + \mathbf{h}$$

we can also define an action on the *right* of H_c on Q_c by

[63] $$\mathrm{R}_{\mathbf{h}}\mathbf{q} = \mathbf{q}\,\mathbf{h} = \mathbf{q} + \left(1+\tfrac{\mathbf{q}^2}{c}\right)\mathbf{h}$$

In the notation of chapter 3, we set $\mathbf{u}_1(\mathbf{q}) = \mathbf{e}_1$ and $\mathbf{u}_2(\mathbf{q}) = \frac{\mathbf{q}^1}{c}\mathbf{e}_1 + \left(1+\frac{\mathbf{q}^2}{c}\right)\mathbf{e}_2$ in the definition of $\mathbf{X}_1$ and $\mathbf{X}_2$ respectively. It is then a straightforward computation to check that these are indeed invariant under the right action [63]

[64] $$\{(\mathrm{R}_{\mathbf{h}})_*\mathbf{X}_a\}\big|_{\mathbf{qh}} = \{\iota_{\mathbf{X}_a}d\mathbf{q}\;(\partial_{\mathbf{q}}\mathrm{R}_{\mathbf{h}})\}\big|_{\mathbf{q}}\,\partial_{\mathbf{q}} = \iota_{\mathbf{X}_a}\left(d\mathbf{q} + \tfrac{1}{c}d\mathbf{q}^2 \otimes \mathbf{h}\right)\partial_{\mathbf{q}} =$$
$$= \iota_{\mathbf{X}_a}\left(d\mathbf{q} + \tfrac{1}{c}d\mathbf{q}^2 \otimes \mathbf{h}\right)\partial_{\mathbf{q}} = \left(\mathbf{u}_a + \tfrac{\mathbf{u}_a^2}{c}\mathbf{h}\right)\partial_{\mathbf{q}} = \mathbf{X}_a\big|_{\mathbf{qh}}$$

This result is evidently true for a=1, and for a=2 the last equality follows from the identity $\left(1+\frac{(\mathbf{qh})^2}{c}\right) = \left(1+\frac{\mathbf{q}^2}{c}\right)\left(1+\frac{\mathbf{h}^2}{c}\right)$. Equation [64] reflects the fact that the $\mathbf{X}_a$ are the fundamental vector fields associated with the Lie algebra $\mathcal{H}_c \cong T_e H_c$ and the left action of H_c on Q_c by the correspondence

$$\begin{aligned} \nu : \mathcal{H}_c &\longrightarrow TQ_c \\ \xi &\mapsto \nu(\xi) = \mathbf{X}_\xi \end{aligned} \tag{[65]}$$

where

$$\mathbf{X}_\xi|_{q'} = \frac{d}{dt}\Big|_{t=0}\{\exp(-t\xi)q'\} = \frac{d}{dt}\Big|_{t=0}\{R_{q'}(\exp(-t\xi))\} = \imath_{\mathbf{X}_\xi|_0}\{dq\,(\partial_q R_{q'})\}\big|_0\,\partial_q = \xi\,(\partial_q R_{q'})\big|_0\,\partial_q \tag{[66]}$$

It is now evident that all the $\mathbf{X}_\xi$ are right-invariant since

$$\begin{aligned} \{(R_h)_*\mathbf{X}_\xi\}\big|_{q'h} &= \{\imath_{\mathbf{X}_\xi} dq\,(\partial_q R_h)\}\big|_{q'}\,\partial_q = \\ &= (\xi\,\partial_q R_{q'})\big|_0(\partial_q R_h)\big|_{q'}\,\partial_q = \xi\,\partial_q(R_h \circ R_{q'})\big|_0\,\partial_q = \\ &= \mathbf{X}_\xi\big|_{q'h} \end{aligned} \tag{[67]}$$

Thus, any right-invariant vector field on Q_c is of the form

$$\mathbf{X}_\xi|_q = \left(\xi + \frac{\xi^2}{c}q\right)\partial_q \tag{[68]}$$

$\mathbf{X}_1$ and $\mathbf{X}_2$ correspond to the cases $\xi = (1,0)$ and $\xi = (0,1)$, respectively.

We now consider the lifting of the right action to the cotangent bundle T^*Q_c. This will be denoted by $\tilde{R}_h$ and is the cotangent map to right translation by $h^{-1} = -\left(1+\frac{h^2}{c}\right)^{-1}h$ on Q_c, i.e.,

$$\begin{aligned} T^*R_{h^{-1}} : T^*_q Q_c &\longrightarrow T^*_{qh}\,Q_c \\ \alpha|_q &\mapsto (T^*R_{h^{-1}}(\alpha))\big|_{qh} = dq\,(\partial_q R_{h^{-1}})\big|_{qh}\cdot\alpha|_q \end{aligned} \tag{[69]}$$

where

$$dq\,(\partial_q R_{h^{-1}})\,\partial_q = \left(dq - \frac{1}{c}\left(1+\frac{h^2}{c}\right)^{-1} dq^2 \otimes h\right)\partial_q \tag{[70]}$$

In the case $\alpha = dq \cdot p$ we obtain the desired lift

$$\tilde{R}_h\,[q, p] = [q + \left(1+\frac{q^2}{c}\right)h\,,\ p - \{\frac{1}{c}\left(1+\frac{h^2}{c}\right)^{-1}h\cdot p\}e^2] \tag{[71]}$$

It is now evident that, lifting to T^*Q_c the vector fields $\mathbf{X}_\xi$ defined in [68], we have

[72] $$\tilde{\mathbf{X}}_\xi|_{(\mathbf{q},\,\mathbf{p})} = \left(\xi + \frac{\xi^2}{c}\mathbf{q}\right)\partial_\mathbf{q} - \left\{\partial_\mathbf{q}\left(\left(\xi + \frac{\xi^2}{c}\mathbf{q}\right)\mathbf{p}\right)\right\}\partial_\mathbf{p} =$$
$$= \left(\xi + \frac{\xi^2}{c}\mathbf{q}\right)\cdot\partial_\mathbf{q} - \frac{\xi^2}{c}\mathbf{p}\,\partial_\mathbf{p}$$

and that the vector fields [72] are right invariant relative to the lifted action [71]. Note that this definition of [72] has to be implemented in canonical (Darboux) coordinates, in which case the lifted vector fields are automatically hamiltonian with respect to the right invariant symplectic structure $\omega = d\mathbf{p}\wedge d\mathbf{q}$ ↑ on T^*Q_c. Thus the observable associated to the hamiltonian field $\mathbf{X}_\xi$ is, up to an additive constant

[73] $$P_\xi(\mathbf{q},\mathbf{p}) = \left(\xi + \frac{\xi^2}{c}\mathbf{q}\right)\mathbf{p}$$

The choice of the additive constant is determined by the requirement that, in the $c\to\infty$ limit, we recover the momenta coordinates p_x, p_y, as the hamiltonian functions associated to the basis $\xi_1 = (1,0)$, $\xi_2 = (0,1)$ of the translation algebra. In the future we will set $P_1 = P_{\xi_1}$, $P_2 = P_{\xi_2}$ as given by [73]. These are the hamiltonian functions of $\tilde{\mathbf{X}}_1$, $\tilde{\mathbf{X}}_2$ respectively. In vector notation, we will write

[74] $$\mathbf{P}(\mathbf{q},\mathbf{p}) = \left(1 + \frac{q^2}{c}\right)\mathbf{p} - \frac{p_1}{c}J\cdot\mathbf{q}$$

where $J = \mathbf{e}^1\otimes\mathbf{e}^2 - \mathbf{e}^2\otimes\mathbf{e}^1$

Having produced the first pair of vector fields that satisfy the requirements in alineas a) and b) at the beginning of chapter 4, we proceed to find the remaining pair to complete a parallelization of T^*Q_c, fulfilling the program outlined in chapter 3. We can therefore define

[75] $$\partial_\mathbf{P} = \left(1 + \frac{q^2}{c}\right)^{-1}\left\{1 + \frac{1}{c}\mathbf{e}^1\otimes(J\,\mathbf{q})\right\}\cdot\partial_\mathbf{p}$$

where $1 = \mathbf{e}_1\otimes\mathbf{e}^1 + \mathbf{e}_2\otimes\mathbf{e}^2$. The components of [75] are

[76] $$\partial_{P_1} = \partial_{p_1} - \frac{q^1}{c}\left(1 + \frac{q^2}{c}\right)^{-1}\partial_{p_2}$$

[77] $$\partial_{P_2} = \qquad\qquad \left(1 + \frac{q^2}{c}\right)^{-1}\partial_{p_2}$$

We must note now that these are not hamiltonian with respect to the canonical, right-invariant symplectic structure on T^*Q_c. We thus face the task of finding a solution to the problem delineated in chapter 4. We start by looking at the contraints imposed on the admissible Lie-algebra structure constants by the requirements of equations [57] [60]. Since we are looking for solutions of the type

$$\breve{\mathbf{X}}_{a+n} = \mathbf{A}_{a+n}(\mathbf{q})\cdot\partial_{\mathbf{p}} \tag*{[78]}$$

It is apparent that the following structure constants are already determined to be

$$\begin{cases} \breve{c}_{b\,a}{}^{d} = c_{ba}{}^{d} \\ \breve{c}_{b\,a+n}{}^{d} = \breve{c}_{b+n\,a+n}{}^{d} = 0 \\ \breve{c}_{b\,a}{}^{d+n} = \breve{c}_{b+n\,a+n}{}^{d+n} = 0 \end{cases} \tag*{[79]}$$

From the Jacobi identities [60] we obtain

$$\begin{cases} \frac{1}{c}\breve{c}_{13}{}^{3} = \breve{c}_{23}{}^{4}\,\breve{c}_{14}{}^{3} - \breve{c}_{13}{}^{4}\,\breve{c}_{24}{}^{3} \\ 2\,\breve{c}_{14}{}^{4}\,\breve{c}_{23}{}^{4} = \breve{c}_{13}{}^{4}\left(\frac{1}{c} + \breve{c}_{24}{}^{4} - \breve{c}_{23}{}^{3}\right) \\ 2\,\breve{c}_{13}{}^{3}\,\breve{c}_{24}{}^{3} = \breve{c}_{14}{}^{3}\left(\frac{1}{c} + \breve{c}_{23}{}^{3} - \breve{c}_{24}{}^{4}\right) \\ \breve{c}_{13}{}^{3} = -\,\breve{c}_{14}{}^{4} \end{cases} \tag*{[80]}$$

These, together with the equations resulting from [68]

$$\begin{cases} \breve{c}_{13}{}^{3}A_{32} + \breve{c}_{13}{}^{4}A_{42} = \left(\breve{c}_{23}{}^{3} - \frac{1}{c}\right)A_{31} + \breve{c}_{23}{}^{4}A_{41} \\ \breve{c}_{14}{}^{3}A_{32} + \breve{c}_{14}{}^{4}A_{42} = \breve{c}_{24}{}^{3}A_{31} + \left(\breve{c}_{24}{}^{4} - \frac{1}{c}\right)A_{41} \end{cases} \tag*{[81]}$$

evaluated at some initial condition, determine at most six equations constraining eight unknown structure constants. We thus see that, in order to have a well defined solution, we need to impose two extra conditions on the possible solutions to the system of partial differential equations which is obtained from [57]

[82]
$$\begin{cases} \partial_{q^1} A_{31} = \check{c}_{13}{}^3 A_{31} + \check{c}_{13}{}^4 A_{41} \\ \partial_{q^1} A_{41} = \check{c}_{14}{}^3 A_{31} + \check{c}_{14}{}^4 A_{41} \\ \partial_{q^1} A_{32} = \check{c}_{23}{}^3 A_{31} + \check{c}_{23}{}^4 A_{41} \\ \partial_{q^1} A_{42} = \check{c}_{24}{}^3 A_{31} + \check{c}_{24}{}^4 A_{41} \end{cases}$$

[83]
$$\begin{cases} \partial_{q^2} A_{31} = \left(1+\frac{q^2}{c}\right)^{-1}\left\{\check{c}_{13}{}^3\left(A_{32}-\frac{q^1}{c}A_{31}\right) + \check{c}_{13}{}^4\left(A_{42}-\frac{q^1}{c}A_{41}\right)\right\} \\ \partial_{q^2} A_{41} = \left(1+\frac{q^2}{c}\right)^{-1}\left\{\check{c}_{14}{}^3\left(A_{32}-\frac{q^1}{c}A_{31}\right) + \check{c}_{14}{}^4\left(A_{42}-\frac{q^1}{c}A_{41}\right)\right\} \\ \partial_{q^2} A_{32} = \left(1+\frac{q^2}{c}\right)^{-1}\left\{\check{c}_{23}{}^3\left(A_{32}-\frac{q^1}{c}A_{31}\right) + \check{c}_{23}{}^4\left(A_{42}-\frac{q^1}{c}A_{41}\right)\right\} \\ \partial_{q^2} A_{42} = \left(1+\frac{q^2}{c}\right)^{-1}\left\{\check{c}_{24}{}^3\left(A_{32}-\frac{q^1}{c}A_{31}\right) + \check{c}_{24}{}^4\left(A_{42}-\frac{q^1}{c}A_{41}\right)\right\} \end{cases}$$

We now specify what type of solution we want. The most natural assumption is to make the 'ansatz' that the basis vector $\breve{\mathbf{X}}_{a+n}$ is collinear with the vector ∂_{p_a}, i.e. $A_{41} = A_{32} = 0$, and that at the origin $A_{31}(0) = A_{42}(0) = -1$. This immediately determines $\check{c}_{14}{}^3 = \check{c}_{23}{}^3 = \check{c}_{23}{}^4 = \check{c}_{14}{}^4 = 0$, which follow from the differential equations above for A_{41} and A_{32}. From [80] then follows that in addition $\check{c}_{24}{}^3 = 0$, and $\check{c}_{13}{}^4 = -\frac{1}{c}$. Also, from the second and fourth of equations [79] we conclude that $\check{c}_{24}{}^4 = -\frac{1}{c}$ and $\check{c}_{13}{}^3 = 0$.

Finally, integration of the system above with this particular choice yields $A_{31}(q) = A_{42}(q) = \lambda\left(1+\frac{q^2}{c}\right)^{-1}$, with λ an arbitrary constant to be fixed by initial conditions, and which we will henceforth set to be $\lambda = -1$ with no loss of generality. We can now exhibit the full Lie algebra basis:

[84]
$$\begin{cases} \breve{\mathbf{X}}_1 = \partial_{q^1} \\ \breve{\mathbf{X}}_2 = \frac{q^1}{c}\partial_{q^1} + \left(1+\frac{q^2}{c}\right)\partial_{q^2} - \frac{p_1}{c}\partial_{p_1} - \frac{p_2}{c}\partial_{p_2} \\ \breve{\mathbf{X}}_3 = -\left(1+\frac{q^2}{c}\right)^{-1}\left\{\partial_{p_1} - \frac{q^1}{c}\left(1+\frac{q^2}{c}\right)^{-1}\partial_{p_2}\right\} \\ \breve{\mathbf{X}}_4 = -\left(1+\frac{q^2}{c}\right)^{-2}\partial_{p_2} \end{cases}$$

and the corresponding multiplication table.

[85]

$[\,,\,]$	$\check{\mathbf{X}}_1$	$\check{\mathbf{X}}_2$	$\check{\mathbf{X}}_3$	$\check{\mathbf{X}}_4$
$\check{\mathbf{X}}_1$	0	$\frac{1}{c}\check{\mathbf{X}}_1$	$-\frac{1}{c}\check{\mathbf{X}}_4$	0
$\check{\mathbf{X}}_2$	$-\frac{1}{c}\check{\mathbf{X}}_1$	0	0	$-\frac{1}{c}\check{\mathbf{X}}_4$
$\check{\mathbf{X}}_3$	$\frac{1}{c}\check{\mathbf{X}}_4$	0	0	0
$\check{\mathbf{X}}_4$	0	$\frac{1}{c}\check{\mathbf{X}}_4$	0	0

The observables associated with the basis elements $\check{\mathbf{X}}_3$ $\check{\mathbf{X}}_4$ are therefore:

$$\begin{cases} \mathbf{Q}_1(\mathbf{q}) = \left(1+\frac{\mathbf{q}^2}{c}\right)^{-1}\mathbf{q}^1 \\ \mathbf{Q}_2(\mathbf{q}) = \left(1+\frac{\mathbf{q}^2}{c}\right)^{-1}\mathbf{q}^2 \end{cases} \tag{86}$$

where once again the arbitrary constants have been chosen in such a way that these deform to the expected observables in the flat case limit.

We have thus obtained a Lie algebra parallelization by hamiltonian vector fields of the phase space T^*Q_c. Due to the special physical meaning that we will attach to them, we now change our coordinate system to the one naturally associated to the basis elements of this parallelization through their hamiltonian functions $\{Q_1,Q_2,P_1,P_2\}$, which we denote in the future by **(Q, P)**, where, according to [74] and [86],

$$\begin{cases} \mathbf{Q}(\mathbf{q},\mathbf{p}) = \left(1+\frac{\mathbf{q}^2}{c}\right)^{-1}\mathbf{q} \\ \mathbf{P}(\mathbf{q},\mathbf{p}) = \left(1+\frac{\mathbf{q}^2}{c}\right)\mathbf{p} - \frac{p_1}{c}J\cdot\mathbf{q} \end{cases} \tag{87}$$

This notation reflects our perspective that these should be interpreted as the *position* and *momenta* observables, respectively. This is because, as we will show later, the hamiltonian vector fields associated with **P** generate the (non-commutative) isometric translations in configuration space, whereas those associated with the **Q** generate the isotropy sub-group fixing the points in configuration space, but generate 'boost'-like transformations on the fibers of T^*Q_c. In this new coordinate system, which we will call (left) Hamiltonian coordinates, the basis for the parallelization reads

[88]
$$\begin{cases} \check{\mathbf{X}}_1 = \left(1-\frac{Q^2}{c}\right)\partial_{Q^1} + \frac{P_1}{c}\partial_{P_2} \\ \check{\mathbf{X}}_2 = \left(1-\frac{Q^2}{c}\right)\partial_{Q^2} - \frac{P_1}{c}\partial_{P_1} \\ \check{\mathbf{X}}_3 = \qquad -\left(1-\frac{Q^2}{c}\right)\partial_{P_1} \\ \check{\mathbf{X}}_4 = \qquad -\left(1-\frac{Q^2}{c}\right)\partial_{P_2} \end{cases}$$

In this coordinate system the right invariant symplectic form [50] reads

[89]
$$\omega = \left(1-\frac{Q^2}{c}\right)^{-1} dP\wedge dQ\cdot 1 - \frac{P_1}{2c}\left(1-\frac{Q^2}{c}\right)^{-2} dQ\wedge dQ\, J$$

We now introduce condition (c) (see p 13) which requires that the Lie algebra action generated by the $\check{\mathbf{X}}_a$ and $\check{\mathbf{X}}_{a+n}$ on T^*Q be integrable to a Lie group action on the same manifold. Integrability requires the existence of a set of left- or right invariant 1 forms $\check{\beta}^k$ ($k=1,\dots,2n$), that verify the Maurer-Cartan equations

[90]
$$d\check{\beta}^k = -\frac{1}{2}\check{c}_{rs}{}^k\,\check{\beta}^r\wedge\check{\beta}^s$$

For an integrable Lie algebra parallelization $\wp(Q)$, we can choose the fields $\mathbf{X}_a\in TQ$ to be right-invariant relative to the group action they generate on Q. Then both the lifted vector fields $\check{\mathbf{X}}_a\in T(T^*Q)$ and their dual forms $\check{\beta}^a\in T^*(T^*Q)$ will be right-invariant relative to the lift of this action to T^*Q. It is therefore natural to require of the remaining vector fields $\check{\mathbf{X}}_{a+n}$ to be right invariant relative to the resulting group action on T^*Q. This in turn means that the dual forms $\check{\beta}^{a+n}\in T^*(T^*Q)$ are also right-invariant. Recall that Q_c is a group manifold, and as such possesses an integrable Lie algebra parallelization; thus equations [90] will be verified if we use as Maurer-Cartan forms the dual of the basis [88]:

[91]
$$\begin{cases} \check{\beta}^1 = \left(1-\frac{Q^2}{c}\right)^{-1} dQ^1 \\ \check{\beta}^2 = \left(1-\frac{Q^2}{c}\right)^{-1} dQ^2 \\ \check{\beta}^3 = -\left(1-\frac{Q^2}{c}\right)^{-1} dP_1 - \left(1-\frac{Q^2}{c}\right)^{-2}\frac{P_1}{c}dQ^2 \\ \check{\beta}^4 = -\left(1-\frac{Q^2}{c}\right)^{-1} dP_2 + \left(1-\frac{Q^2}{c}\right)^{-2}\frac{P_1}{c}dQ^1 \end{cases}$$

It is straightforward to check that this is indeed the case, i.e

$$[92]\qquad \begin{cases} d\check{\beta}^1 = -\frac{1}{c}\,\check{\beta}^1\wedge\check{\beta}^2 \\ d\check{\beta}^2 = 0 \\ d\check{\beta}^3 = 0 \\ d\check{\beta}^4 = \frac{1}{c}\left(\check{\beta}^1\wedge\check{\beta}^3 + \check{\beta}^2\wedge\check{\beta}^4\right) \end{cases}$$

Therefore, by integrating the differential equations that express the invariance of the Maurer-Cartan forms $\check{\beta}^\kappa$ under the group action $\Lambda.\ G_c\times T^*Q_c \longrightarrow T^*Q_c$

$$[93]\qquad \overset{*}{\Lambda}_g\check{\beta}^\kappa = \check{\beta}^\kappa \qquad\qquad (\forall\, g\in G_c,\ \kappa = 1,2,3,4)$$

we will obtain a 4-dimensional group G_c which will turn out to be $H_c\times\mathbb{R}^2$ with multiplication law (in a coordinate system where $g = (A, B)$)

$$[94]\qquad (A, B)(A', B') = \left(A + \left(1-\frac{A^2}{c}\right)A',\ B + \left(1-\frac{A^2}{c}\right)B' - \frac{B_1}{c}\,J\cdot A'\right)$$

acting canonically on T^*Q_c on the left via

$$[95]\quad \Lambda_g(Q,P) = (A, B)(Q,P) = \left(A + \left(1-\frac{A^2}{c}\right)Q,\ B + \left(1-\frac{A^2}{c}\right)P - \frac{B_1}{c}\,J\,Q\right)$$

To see this we start by writing [93] in explicit form. If we set $\check{\beta}^\kappa = dQ\,a^\kappa + dP\,b^\kappa$, then

$$[96]\qquad \overset{*}{\Lambda}_g\check{\beta}^\kappa = d\left(\Lambda_g^*Q\right)\Lambda_g^*a^\kappa + d\left(\Lambda_g^*P\right)\Lambda_g^*b^\kappa = dQ\,a^\kappa + dP\,b^\kappa$$

Taking into consideration [91], this can be written explicitly as

$$\begin{cases} \left(1-\frac{\Lambda_g^*Q^2}{c}\right)^{-1} d\left(\Lambda_g^*Q^1\right) = \left(1-\frac{Q^2}{c}\right)^{-1} dQ^1 \\ \left(1-\frac{\Lambda_g^*Q^2}{c}\right)^{-1} d\left(\Lambda_g^*Q^2\right) = \left(1-\frac{Q^2}{c}\right)^{-1} dQ^2 \\ \frac{\Lambda_g^*P_1}{c}\left(1-\frac{\Lambda_g^*Q^2}{c}\right)^{-2} d\left(\Lambda_g^*Q^2\right) + \left(1-\frac{\Lambda_g^*Q^2}{c}\right)^{-1} d\left(\Lambda_g^*P_1\right) = \frac{P_1}{c}\left(1-\frac{Q^2}{c}\right)^{-2} dQ^2 + \left(1-\frac{Q^2}{c}\right)^{-1} dP_1 \\ \frac{\Lambda_g^*P_1}{c}\left(1-\frac{\Lambda_g^*Q^2}{c}\right)^{-2} d\left(\Lambda_g^*Q^1\right) - \left(1-\frac{\Lambda_g^*Q^2}{c}\right)^{-1} d\left(\Lambda_g^*P_2\right) = \frac{P_1}{c}\left(1-\frac{Q^2}{c}\right)^{-2} dQ^1 - \left(1-\frac{Q^2}{c}\right)^{-1} dP_2 \end{cases}$$

We begin by integrating the second equation of this Pfaff system. This yields

[97]
$$\left(1-\frac{A_g^* Q^2}{c}\right)=\lambda^2(g)\left(1-\frac{Q^2}{c}\right)$$

where $\lambda^2(g)$ is determined by initial conditions. Obviously $\lambda^2(e)=1$, and

[98]
$$A_g^* Q^2=\lambda^2(g)\,Q^2+c\left(1-\lambda^2(g)\right)$$

inserting [98] into the first equation of the Pfaff system we get

[99]
$$A_g^* Q^1=\lambda^2(g)\,Q^1+\lambda^1(g)$$

with $\lambda^1(e)=0$. This in turn will be recognized as the action of the isometry group H_c on the configuration space Q_c since we can rewrite with no loss of generality

[100]
$$\begin{cases}\lambda^1(g)=A^1(g)\\ \lambda^2(g)=\left(1-\frac{A^2(g)}{c}\right)\end{cases}$$

The third equation of the Pfaff system relates exact 1-forms, and its solution is immediately determined to be

[101]
$$A_g^* P_1\left(1-\frac{A_g^* Q^2}{c}\right)^{-1}=P_1\left(1-\frac{Q^2}{c}\right)^{-1}+\lambda^3(g)$$

with $\lambda^3(e)=0$ This can be further simplified to

[102]
$$A_g^* P_1=P_1\left(1-\frac{A^2(g)}{c}\right)+\lambda^3(g)\left(1-\frac{A^2(g)}{c}\right)\left(1-\frac{Q^2}{c}\right)$$

and if we denote $\lambda^3(g)\left(1-\frac{A^2(g)}{c}\right)=B_1(g)$,

[103]
$$A_g^* P_1=P_1\left(1-\frac{A^2(g)}{c}\right)+\left(1-\frac{Q^2}{c}\right)B_1(g)$$

We now use these in the last equation of the Pfaff system, and obtain

[104]
$$A_g^* P_2=P_2\left(1-\frac{A^2(g)}{c}\right)+\frac{Q^1}{c}B_1(g)+\lambda^4(g)$$

with $\lambda^4(e)=0$. Redesignating $\lambda^4(g)$ by $B_2(g)$, we finally obtain expression [95], and demanding that it be a group *action* is enough to derive the multiplication law [94]

Note that, due to our choice of integration constants, the group identity is parameterized by $(\mathbf{A}, \mathbf{B}) = (\mathbf{0}, \mathbf{0})$. In fact, it is easy to see from expressions [94] and [95] that there is a one-to-one correspondence between the group manifold G_c and the phase space T^*Q_c, and the group action is identifiable with the group multiplication.

If we denote the Lie algebra of G_c by $\mathcal{G}_c$, i.e. the vector space generated by the right invariant vector fields [88], then its derived algebra is $\mathcal{G}_{c[1]} = [\mathcal{G}_c, \mathcal{G}_c] = \{\tilde{\mathbf{X}}_1, \tilde{\mathbf{X}}_4\}$, and therefore $\mathcal{G}_{c[2]} = [\mathcal{G}_{c[1]}, \mathcal{G}_{c[1]}] = \{0\}$. The group G_c is thus shown to be *solvable*, and is actually *exponential*, i.e. there is a smooth bijection between $\exp(\mathcal{G}_c)$ and G_c. In fact, the one-parameter subgroups through the origin of G_c generated by the vector field

[105] $$\tilde{\mathbf{X}}_{[\mathbf{a},\mathbf{b}]} = \sum_{e=1}^{2} \{ a^e \tilde{\mathbf{X}}_e + b^e \tilde{\mathbf{X}}_{e+2} \} \in \mathcal{G}_c$$

are of the form

[106] $$\exp(-t\,\tilde{\mathbf{X}}_{[\mathbf{a},\mathbf{b}]}) = (\mathbf{A}(t), \mathbf{B}(t))$$

where

[107] $$\begin{cases} \mathbf{A}(t) = \frac{c}{a^2}\left(1 - e^{-\frac{a^2 t}{c}}\right)\mathbf{a} \\ \mathbf{B}(t) = \frac{c}{a^2}\left(1 - e^{-\frac{a^2 t}{c}}\right)\mathbf{b} + \frac{b_3}{a^2}\left\{ t\, e^{-\frac{a^2 t}{c}} - \frac{c}{a^2}\left(1 - e^{-\frac{a^2 t}{c}}\right)\right\} J\,\mathbf{a} \end{cases}$$

These equations show that the identity is connected with any point in G_c through a one-parameter subgroup, and that the action Λ of G_c on T^*Q_c is free and transitive. Thus G_c is a exponential group. Furthermore, in the $c \to \infty$ limit, G_c deforms as expected to the boost-translation subgroup of the Galilei group acting in $T^*\mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{R}^2$

CHAPTER 6
THE WEYL-HEISENBERG GROUP W_C IN CURVED SPACE

We will note at this point that, when one uses the sympletic formalism to describe classical mechanics on flat, Euclidean space, one encounters a familiar ingredient usually singled out in the connection with the *quantum mechanical* formalism: the Weyl Heisenberg group. In fact, any sympletic manifold of the form $\{T^*\mathbb{R}^n,\ \omega = d\mathbf{p}\wedge d\mathbf{q}\}$ can be viewed as the *coadjoint orbit* of the (2n+1)-dimensional Heisenberg group W_n on the dual of its Lie algebra $\mathcal{W}_n$ –*the* Canonical Commutation Relations (C.C.R.) algebra.

Note further that $T^*\mathbb{R}^n$ is homogeneous relative to $G = \mathbb{R}^{2n}$, acting transitively by the mapping

$$\begin{aligned} A : \ & \mathbb{R}^{2n}\times T^*\mathbb{R}^n \longrightarrow T^*\mathbb{R}^n \\ & (\mathbf{g},\ \mathbf{m}) \ \mapsto A_{\mathbf{g}}(\mathbf{m}) = \mathbf{m} + \mathbf{g} \end{aligned} \qquad [108]$$

In canonical coordinates $\mathbf{m} = (\mathbf{q},\mathbf{p})$, with $\mathbf{p}$ and $\mathbf{q} \in \mathbb{R}^n$ and if we let $\mathbf{g} = (\mathbf{a},\mathbf{b})$, with $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, this action reads

$$A_{\mathbf{g}}(\mathbf{m}) = (\mathbf{q}+\mathbf{a},\ \mathbf{p}+\mathbf{b}) \qquad [109]$$

This action is canonical relative to the natural symplectic structure $\omega = d\mathbf{p}\wedge d\mathbf{q}$, and therefore $G = \mathbb{R}^{2n}$ is a dynamical group for the free dynamics on Euclidean spaces. Also, the infinitesimal generators of this action on $T^*\mathbb{R}^n$ are hamiltonian vector fields $\mathbf{X}\in T(T^*\mathbb{R}^n)$, since from canonicity of their flows one gets $\mathcal{L}_{\mathbf{X}}\omega = 0$. In fact, there is a Lie algebra homomorphism

$$\begin{aligned} \sigma : \ & \mathcal{G} \longrightarrow \mathcal{H}am(T^*\mathbb{R}^n) \\ & \xi \ \mapsto \sigma(\xi) = \mathbf{X}_{\xi} \end{aligned} \qquad [110]$$

where $\mathcal{G} \cong T_e\mathbb{R}^{2n}$ and

[111] $$\mathbf{X}_{\boldsymbol{\xi}}|_m = \frac{d}{dt}\Big|_{t=0}\left\{ \Lambda_{\exp(-t\boldsymbol{\xi})}(m) \right\}$$

is the *fundamental* vector field associated with the Lie algebra generator $\boldsymbol{\xi}$

In [110], $\mathcal{H}am\left(T^*\mathbb{R}^n\right)$ denotes the Lie sub-algebra of $\mathcal{C}an\left(T^*\mathbb{R}^n\right)$ — the algebra of infinitesimal generators of canonical transformations on $T^*\mathbb{R}^n$ with Lie bracket as anti-symmetric product — whose elements $\mathbf{X} \in \mathcal{H}am\left(T^*\mathbb{R}^n\right)$ are strongly hamiltonian vector fields, i e., those for which the closed 1-forms $\vartheta_{\mathbf{X}} = i_{\mathbf{X}}\omega$ are exact $\vartheta_{\mathbf{X}} = -df$, for some $f \in \mathcal{C}^{\infty}_{\mathbb{R}}\left(T^*\mathbb{R}^n\right)$

Again, in local coordinates, if we set $\boldsymbol{\xi} = (\mathbf{a}\ \mathbf{b}) \in \mathcal{G} = \mathbb{R}^{2n}$, [111] reads

[112] $$\mathbf{X}_{\boldsymbol{\xi}} = \mathbf{a}\,\partial_q + \mathbf{b}\,\partial_p$$

since, in this case, the exponential map is the identity map. It is also apparent that

[113] $$\vartheta_{\mathbf{X}_{\boldsymbol{\xi}}} = -d(\mathbf{a}\cdot\mathbf{p} - \mathbf{b}\cdot\mathbf{q})$$

which suggests that there is a correspondence

[114] $$\begin{aligned} \hat{\mu}_0 : \mathcal{G} &\longrightarrow \mathcal{C}^{\infty}_{\mathbb{R}}\left(T^*\mathbb{R}^n\right) \\ \boldsymbol{\xi} &\mapsto \hat{\mu}_0(\boldsymbol{\xi}) = [f_{\boldsymbol{\xi}}] \end{aligned}$$

between each element $\boldsymbol{\xi} = (\mathbf{a}, \mathbf{b}) \in \mathcal{G}$ and a class of functions (*observables*)

[115] $$f_{(\boldsymbol{\xi},\kappa)}(\mathbf{q}\,\mathbf{p}) = \mathbf{a}\cdot\mathbf{p} - \mathbf{b}\cdot\mathbf{q} + \kappa \in \mathcal{C}^{\infty}_{\mathbb{R}}\left(T^*\mathbb{R}^n\right)$$

with $\kappa \in \mathbb{R}$ an arbitrary, additive constant

Moreover, we know that there exists a map

[116] $$\begin{aligned} \chi : \mathcal{C}^{\infty}_{\mathbb{R}}\left(T^*\mathbb{R}^n\right) &\longrightarrow \mathcal{H}am\left(T^*\mathbb{R}^n\right) \\ f &\mapsto \chi(f) = \mathbf{X}_f \end{aligned}$$

where $\mathbf{X}_f$ is defined by the solution of the exterior equation

[117] $$i_{\mathbf{X}_f}\omega = -df$$

In canonical coordinates

[118] $$\mathbf{X}_f = -(\partial_{\mathbf{p}} f)\,\partial_{\mathbf{q}} + (\partial_{\mathbf{q}} f)\,\partial_{\mathbf{p}}$$

All this can be subsumed with the diagram

[119] $$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r] & C^{\infty}_{\mathbb{R}}(T^*\mathbb{R}^n) \arrow[r, "\chi"] & \mathcal{H}am(T^*\mathbb{R}^n) \arrow[r] & 0 \\ & & & \mathcal{G} \arrow[ul, "\hat{\mu}_0"] \arrow[u, "\sigma"'] & \end{tikzcd}$$

in which the upper line represents an exact sequence of algebras —taking into consideration that $C^{\infty}_{\mathbb{R}}(T^*\mathbb{R}^n)$ is a Lie algebra with respect to the Poisson bracket

[120] $$\{f_1, f_2\} = \mathbf{X}_{f_1}(f_2) = (\partial_{\mathbf{q}} f_1)\,(\partial_{\mathbf{p}} f_2) - (\partial_{\mathbf{p}} f_1)\,(\partial_{\mathbf{q}} f_2)$$

Note however that $\hat{\mu}_0$ is not a Lie algebra homomorphism in general. All one can say is that

[121] $$-\mathrm{d}\{f_{\xi}, f_{\zeta}\} = -\mathrm{d}(\iota_{\mathbf{X}_{\zeta}} \mathrm{d}f_{\xi}) = \mathrm{d}(\iota_{\mathbf{X}_{\xi}}(\iota_{\mathbf{X}_{\zeta}}\omega)) = \iota_{[\mathbf{X}_{\zeta}, \mathbf{X}_{\xi}]}\omega =$$
$$-\iota_{\mathbf{X}_{[\xi,\zeta]}}\omega = -\mathrm{d}f_{[\xi,\zeta]}$$

i.e., there is a constant $\beta(\xi, \zeta) \in \mathbb{R}$ such that

[122] $$\hat{\mu}_0([\xi, \zeta]) = \{\hat{\mu}_0(\xi),\ \hat{\mu}_0(\zeta)\} + \beta(\xi, \zeta)$$

Due to the anti-symmetry of β, which follows promptly from [122], and using Jacobi's identities, it becomes apparent that $\beta\colon \mathcal{G} \times \mathcal{G} \longrightarrow \mathbb{R}$ is indeed a Lie algebra co-cycle, and its obstruction to the homomorphic character of $\hat{\mu}_0$ can be removed by working with a *central extension* $\mathcal{W}_n$ of the Lie algebra $\mathcal{G}$ by $\mathbb{R}$ (see, for example, the recent review by Tuynman and Wiegerinck [Tuy]), determined by setting the Lie algebra structure to obey

[123] $$[\xi_i, \xi_j] = c_{ij}^{\ k}\,\xi_k + \beta(\xi_i, \xi_j)\,\xi_R \qquad [\xi_i, \xi_R] = 0$$

where c_{ij}^{k} are the structure constants for the non-extended Lie algebra $\mathcal{G}$, ξ_i $(i=1,2,\ldots,n)$ a basis of $\mathcal{G}$ and $\xi_{\mathbb{R}}$ the generator associated with the extension $\mathbb{R}$. Since $\mathcal{G}$ is abelian in the flat case, it follows from this that

$$[124] \qquad \beta(\xi, \xi') = -\{\hat{\mu}_0(\xi), \hat{\mu}_0(\xi')\} = \mathbf{b}\cdot\mathbf{a}' - \mathbf{a}\cdot\mathbf{b}'$$

It is then apparent that $\mathcal{W}_n$ is in fact the Lie algebra of the familiar Heisenberg group of dimension 2n+1 (see for example [Emc]), here denoted W_n.

For this extended Lie algebra $\mathcal{W}_n$ we can then define a Lie algebra homomorphism, called a co-momentum map, $\hat{\mu}: \mathcal{W}_n \longrightarrow C^{\infty}_{\mathbb{R}}(T^*\mathbb{R}^n)$, by extending the definition of $\hat{\mu}_0$ to map $\xi_{\mathbb{R}}$ to the constant $f_{\xi_{\mathbb{R}}} = 1$. This in turn determines a *momentum mapping* μ for the Lie algebra action on M, by setting

$$[125] \qquad \begin{aligned} \mu: \ & M \longrightarrow \mathcal{W}_n^* \\ & m \mapsto \mu_m: \mathcal{W}_n \longrightarrow \mathbb{R} \\ & \qquad\qquad (\xi, \kappa) \mapsto \mu_m(\xi, \kappa) = f_{(\xi,\kappa)}(m) \end{aligned}$$

(Note that a momentum map for a *symplectic* action $\sigma: \mathcal{G} \longrightarrow \mathcal{C}an(M,\omega)$ only exists if each $\sigma(\xi) = X_\xi$ is globally hamiltonian, i.e., belongs to $\mathcal{H}am(M,\omega)$. This always happens if either M is simply-connected, G is semi simple or $\mathcal{G} = [\mathcal{G}, \mathcal{G}]$.)

For the flat case we have been considering in this chapter, $f_{(\xi,\kappa)}(m) = \mathbf{a}\cdot\mathbf{p} - \mathbf{b}\cdot\mathbf{q} + \kappa$; thus a convenient momentum map is of the form

$$[126] \qquad \mu_{(\mathbf{q},\mathbf{p})} = (\mathbf{p}, -\mathbf{q}, 1) \in \mathcal{W}_n^*$$

which is a consequence of [115] insofar as we agree to write $(\xi,\kappa) = (\mathbf{a}, \mathbf{b}, \kappa)$ for $(\xi,\kappa) \in \mathcal{W}_n$, $\kappa \in \mathbb{R}$.

We now focus on the determination of a momentum map for the symplectic action of G_c on T^*Q_c. We recall here that this presupposes the existence of a co-momentum map which establishes an isomorphism between the algebra obtained via a central extension of the solvable dynamical algebra $\mathcal{G}_c$ and the Heisenberg-type Lie algebra generated by the basic observables $(\mathbf{Q}, \mathbf{P})$, whose Poisson brackets in $\mathcal{C}^{\infty}_{\mathbb{R}}(\mathrm{T}^*\mathrm{Q}_c)$ are

$$[127] \qquad \begin{cases} \{Q^1, P_1\} = \tilde{X}_3(P_1) = \frac{1}{c}Q^2 - 1 \\ \{Q^2, P_2\} = \tilde{X}_4(P_2) = \frac{1}{c}Q^2 - 1 \\ \{P_1, P_2\} = \tilde{X}_1(P_2) = \frac{1}{c}P_1 \\ \{Q^1, Q^2\} = \{Q^1, P_2\} = \{Q^2, P_1\} = 0 \end{cases}$$

As in the flat case, this is a realization of the Lie algebra $\mathcal{W}_c$ obtained by an abelian extension of the dynamical algebra $\mathcal{G}_c$ by $\mathbb{R}$, in such a way that the new generator ξ_0, in the center of $\mathcal{W}_c$, acts trivially on $\mathrm{T}^*\mathrm{Q}_c$ and its image under the co-momentum map $\hat{\mu}$: $\mathcal{W}_c \longrightarrow \mathcal{C}^{\infty}_{\mathbb{R}}(\mathrm{T}^*\mathrm{Q}_c)$ is $\hat{\mu}(\xi_0) = f_{\xi_0} = 1$. The commutation relations of this Weyl Heisenberg algebra are now

$$[128] \qquad \begin{cases} [\xi_1, \xi_2] = \frac{1}{c}\xi_1 \\ [\xi_1, \xi_3] = \xi_0 - \frac{1}{c}\xi_4 \\ [\xi_2, \xi_4] = \xi_0 - \frac{1}{c}\xi_4 \\ [\xi_1, \xi_4] = [\xi_2, \xi_3] = [\xi_0, \xi_i] = 0 \end{cases}$$

where we have used the abbreviated notation $\xi_1 = \xi_{P_1}$, $\xi_2 = \xi_{P_2}$, $\xi_3 = \xi_{Q^1}$ and $\xi_4 = \xi_{Q^2}$.

Integration of this Lie algebra in order to obtain the corresponding Weyl-Heisenberg group W_c cannot be modeled this time on its realization as hamiltonian vector fields on $\mathrm{T}^*\mathrm{Q}_c$, as the image of ξ_0 under the Lie algebra homomorphism σ: $\mathcal{W}_c \longrightarrow \mathcal{H}am(\mathrm{T}^*\mathrm{Q}_c)$ is $\sigma(\xi_0) = \tilde{X}_{\xi_0} = 0$. Instead, we use the fact that we can find *canonical coordinates of the first kind* $\chi = (\chi^0, \chi^1, \chi^2, \chi^3, \chi^4)$ on W_c (i.e., coordinates such that $\chi(t) = \exp(t\xi) = t\xi$ are the coordinates of a one-parameter group, for $\xi \in \mathcal{W}_c$ fixed) relative to which we can define a set of functions $\mathrm{B}^{\kappa}_{\sigma}(\chi)$, $(\kappa, \sigma = 0, \ldots, 4)$, differentiable to the second order at least, such that the 1 forms $\beta^{\kappa} = \mathrm{B}^{\kappa}_{\sigma}(\chi)d\chi^{\sigma}$ satisfy the integrability conditions

[129]
$$\begin{cases} d\beta^0 = -\left(\beta^1\wedge\beta^3 + \beta^2\wedge\beta^4\right) \\ d\beta^1 = -\frac{1}{c}\beta^1\wedge\beta^2 \\ d\beta^2 = 0 \\ d\beta^3 = 0 \\ d\beta^4 = \frac{1}{c}\left(\beta^1\wedge\beta^3 + \beta^2\wedge\beta^4\right) \end{cases}$$

i.e., the Maurer-Cartan equations relative to the structure constants $C^\kappa_{\sigma\rho} = \breve{C}_{\sigma\rho}{}^\kappa$ (for σ, ρ and $\kappa \neq 0$), $C^0_{13} = C^0_{24} = 1$ and all other zero The advantage of this approach is that there is a canonical expression for the functions $B^\kappa_\sigma(\chi)$ in this setting If $B^\kappa_\sigma(t,\chi)$ is a solution to the initial value problem

[130]
$$\partial_t\left(B^\kappa_\sigma(t,\chi)\right) = \delta^\kappa_\sigma - C^\kappa_{\upsilon\rho}\,\chi^\upsilon B^\rho_\sigma(t,\chi) \qquad \text{with } B^\kappa_\sigma(0,\chi) = 0$$

then it is known (see e.g [Sat]) that $B^\kappa_\varepsilon(\chi) = B^\kappa_\sigma(1,\chi)$ is a solution of the integrability conditions [129] Once we get the solution of equations [130] we then have to integrate the over-determined system of differential equations

[131]
$$\Lambda^*_\varepsilon\beta^\kappa = \beta^\kappa$$

to obtain the multiplication law Λ: $W_C\times W_C \longrightarrow W_C$ We can perform better the integration of [130] if we group these equations in vector form

[132]
$$\begin{cases} \partial_t \mathbf{B}^0 = \mathbf{e}^0 + \chi^3\mathbf{B}^1 + \chi^4\mathbf{B}^2 - \chi^1\mathbf{B}^3 - \chi^2\mathbf{B}^4 \\ \partial_t \mathbf{B}^1 = \mathbf{e}^1 + \frac{1}{c}\chi^2\mathbf{B}^1 - \frac{1}{c}\chi^1\mathbf{B}^2 \\ \partial_t \mathbf{B}^2 = \mathbf{e}^2 \\ \partial_t \mathbf{B}^3 = \mathbf{e}^3 \\ \partial_t \mathbf{B}^4 = \mathbf{e}^4 - \frac{1}{c}\chi^3\mathbf{B}^1 - \frac{1}{c}\chi^4\mathbf{B}^2 + \frac{1}{c}\chi^1\mathbf{B}^3 + \frac{1}{c}\chi^2\mathbf{B}^4 \end{cases}$$

where $\mathbf{B}^\kappa = B^\kappa_\sigma(t,\chi)\,\mathbf{e}^\sigma$, so that we can write $\beta^\kappa = d\chi \cdot \mathbf{B}^\kappa|_{t=1}$ The integration of this system is straightforward albeit lengthy, and yields

$$[133] \quad \begin{cases} \mathbf{B}^0 = \mathrm{t}\,\mathbf{e}^0 + \mathrm{c}\,\mathrm{t}\,\mathbf{e}^4 - \mathrm{c}\,\mathbf{B}^4(t,\chi) \\ \mathbf{B}^1 = -\frac{c}{\varkappa^2}\left(1-e^{\frac{t\varkappa^2}{c}}\right)\mathbf{e}^1 + \frac{\chi^1_2}{\varkappa^2}\left\{t+\frac{c}{\varkappa^2}\left(1-e^{\frac{t\varkappa^2}{c}}\right)\right\}\mathbf{e}^2 \\ \mathbf{B}^2 = \mathrm{t}\,\mathbf{e}^2 \\ \mathbf{B}^3 = \mathrm{t}\,\mathbf{e}^3 \\ \mathbf{B}^4 = -\frac{\chi^3_2}{\varkappa^2}\left\{t\,e^{\frac{t\varkappa^2}{c}}+\frac{c}{\varkappa^2}\left(1-e^{\frac{t\varkappa^2}{c}}\right)\right\}\mathbf{e}^1 + \\ \quad + \frac{\chi^1_2}{\varkappa^2}\left[\frac{\chi^3_2}{\varkappa^2}\left\{t\,e^{\frac{t\varkappa^2}{c}}+\frac{c}{\varkappa^2}\left(1-e^{\frac{t\varkappa^2}{c}}\right)\right\}+\left(\frac{\chi^3_2}{\varkappa^2}+\frac{\chi^4}{c}\right)\left\{t+\frac{c}{\varkappa^2}\left(1-e^{\frac{t\varkappa^2}{c}}\right)\right\}\right]\mathbf{e}^2 - \\ \qquad - \frac{\chi^1_2}{\varkappa^2}\left\{t+\frac{c}{\varkappa^2}\left(1-e^{\frac{t\varkappa^2}{c}}\right)\right\}\mathbf{e}^3 - \frac{c}{\varkappa^2}\left(1-e^{\frac{t\varkappa^2}{c}}\right)\mathbf{e}^4 \end{cases}$$

Although these seem too complicated to be of use, we note that, in a sense, we already have some information on how to simplify the following computations. We know of a realization for the sub-algebra of $\mathscr{W}_c$ generated by $\{\xi_{s_1}, \xi_{s_2}, \xi_{s_3}, \xi_{s_4}\}$ and we can find a coordinate system in which their dual 1 forms $\{\beta^1, \beta^2, \beta^3, \beta^4\}$ have the simpler expressions [91]. This is accomplished by the transformation

$$[134] \quad \begin{cases} \chi^1(\mathbf{Q},\mathbf{P}) = -\mathbf{Q}^1\, \hbar(\mathbf{Q}^2) \\ \chi^2(\mathbf{Q},\mathbf{P}) = -\mathbf{Q}^2\, \hbar(\mathbf{Q}^2) \\ \chi^3(\mathbf{Q},\mathbf{P}) = \mathbf{P}_1\, \hbar(\mathbf{Q}^2) + \mathbf{Q}^2\,\mathbf{P}_1\, \hbar'(\mathbf{Q}^2) \\ \chi^4(\mathbf{Q},\mathbf{P}) = \mathbf{P}_2\, \hbar(\mathbf{Q}^2) - \mathbf{Q}^1\,\mathbf{P}_1\, \hbar'(\mathbf{Q}^2) \end{cases}$$

where

$$[135] \quad \hbar(\mathrm{Q}^2) = \frac{c}{\mathrm{Q}^2}\, \ln\left(1-\frac{\mathrm{Q}^2}{c}\right) \qquad \text{and} \qquad \hbar' = \frac{d\hbar}{d\mathrm{Q}^2}$$

Therefore we need only concern ourselves with integration of equation [131] for $\varkappa = 0$ in which we have explicitly

$$[136] \quad \beta^0 = d\chi^0 + c\, d\chi^4 - c\,\beta^4$$

and

$$[137] \quad d\left(\Lambda^*_c\chi^0 + c\,\Lambda^*_c\chi^4\right) - c\,\Lambda^*_c\beta^4 = d\left(\chi^0 + c\,\chi^4\right) - c\,\beta^4$$

Since β^4 has no $\mathbf{d}\chi^0$ component and really depends only on the $(\mathbf{Q},\mathbf{P})$ coordinates, we have $\Lambda_c^*\beta^4 - \beta^4 = 0$ (recall that the 1-forms $\{\beta^1,\beta^2,\beta^3,\beta^4\}$ are Maurer Cartan forms for the subgroup G_c generated by the sub algebra $\mathcal{G}_c \subset \mathcal{W}_0^c$, with multiplication law [94]); we finally solve for the extended group multiplication law Λ_c by integrating equation [137] at $x = \{\theta, (\mathbf{Q},\mathbf{P})\}$ with initial conditions $x_0 = \{\theta_0, (\mathbf{A},\mathbf{B})\}$, and get

[138] $$\chi^0\big(\Lambda_c(x_0,x)\big) = \theta_0 + \theta + c\left[\chi^4(\mathbf{A},\mathbf{B}) + \chi^4(\mathbf{Q},\mathbf{P}) - \chi^4\big((\mathbf{A},\mathbf{B})(\mathbf{Q},\mathbf{P})\big)\right]$$

where $(\mathbf{A},\mathbf{B})(\mathbf{Q},\mathbf{P})$ is the product [95], and for which the functions $\chi^\kappa\big((\mathbf{A},\mathbf{B})(\mathbf{Q},\mathbf{P})\big)$ provide the solutions of the remaining equations [131] with $\kappa \neq 0$.

It is interesting to observe that, although the coordinate transformation [134] diverges in the flat case limit $c \to \infty$, expression [138] behaves correctly as can be seen from

[139] $$\begin{cases} \lim\limits_{c\to\infty} \hbar(Q^2) = -1 \\ \lim\limits_{c\to\infty} c\,\hbar(Q^2) = -\frac{1}{2} \\ \lim\limits_{c\to\infty} c\left[\hbar((AQ)^2) - \hbar(Q^2)\right] = -\frac{1}{2}A^2 \end{cases}$$

and the careful handling of the expansion of the last term of the r.h.s. of [138], which leads to

[140] $$\lim_{c\to\infty} \chi^0\big(\Lambda_c(x_0,x)\big) = \theta_0 + \theta + \tfrac{1}{2}\left[\mathbf{Q}\cdot\mathbf{B} - \mathbf{P}\cdot\mathbf{A}\right]$$

This is a curious phenomenon that is linked to the fact that the last term of the r.h.s. of [138] is a trivial cocycle in the zero class of the second cohomology of the curved *Weyl Heisenberg* group, whereas the corresponding term in [140] is non-trivial: in fact, for the flat *Heisenberg* group this is the mass 1 cocyle [Bar]. We recall that, given a Lie group G, an abelian group E and a representation $\rho: G \longrightarrow \mathcal{E}nd(E)$, we can define, for every integer $k > 0$, the space of *non homogeneous k-cochains* of G with values in E, as the space $C_k(G,E)$ of differentiable mappings $\vartheta: G^k \longrightarrow E$. In the case $k=0$, the space $C_0(G,E)$ of non-homogeneous 0-cochains is identified with E. The graded vector space $\mathcal{C}(G,E) = \bigoplus_{k=0}^{\infty} C_k(G,E)$ with the *coboundary* operator $\delta \in \mathcal{E}nd\big(\mathcal{C}(G,E)\big)$ defined as

$$[141] \qquad \delta[\vartheta](g_1,g_2,\dots,g_{k+1}) = \rho(g_1)\vartheta(g_2,g_3,\dots,g_{k+1}) + \sum_{i=1}^{k}(-1)^i\,\vartheta(g_1,g_2,\dots,g_i g_{i+1},g_{i+2},\dots,g_{k+1}) + (-1)^{k+1}\,\vartheta(g_1,g_2,\dots,g_k)$$

is called a *complex of non homogeneous cochains*. The cohomology of the Lie *group* G associated with the representation ρ is the cohomology of this complex. Any cochain whose coboundary is nul is called a *cocycle*, and the identity $\delta\circ\delta = 0$ implies that all coboundaries are cocycles but not *vice versa*, the quotient set of all k-cocycles modulo k-coboundaries is the k-th cohomology group $H_k(G,E)$.

The cocycles in $H_2(G,E)$ classify the possible inequivalent central extensions of a group G, and in the case $E = U(1)$ these are also relevant for the determination of the unitary *ray* representations, or projective representations of G. These are realizations of G (or a neighborhood of the identity e thereof) as unitary operators U on some Hilbert space verifying the conditions

$$[142] \qquad \begin{cases} U_e = 1 \\ U_g U_{g'} = \omega(g,g')U_{gg'} \end{cases}$$

where $\omega(g,g')$ is a complex number of modulus 1 (the *local factor* of the given ray-representation), g and $g' \in G$. Furthermore $\omega(e,e) = 1$ and associativity of the group law implies that

$$[143] \qquad \omega(g,g')\,\omega(gg',g'') = \omega(g,g'g'')\,\omega(g',g'')$$

Given one such local factor ω, one can get equivalent factors ω', related to the first one by

$$[144] \qquad \omega'(g,g') = \omega(g,g')\,\phi(g)\,\phi(g')\,\phi(gg')^{-1}$$

This factor defines an equivalent ray representation $U'_g = \phi(g)\,U_g$, where $|\phi(g)| = 1$. Oftentimes it is advantageous to deal with ordinary rather than projective representations and since the operators contained in any operator ray are of the form $e^{i\theta}U_g$, $\theta \in \mathbb{R}$, and form a group under multiplication, it proves convenient [Bar] to define a *local exponent* $\xi(g,g')$ which relates to a local factor through the expression

$$\omega(g,g') = e^{i\xi(g,g')} \tag{145}$$

so that

$$(e^{i\vartheta}U_g)(e^{i\vartheta'}U_{g'}) = e^{i(\vartheta+\vartheta'+\xi(g,g'))}U_{gg'} \tag{146}$$

We therefore see that a local exponent defines, at least locally, a *local group extension* of G by $\mathbb{R}$, whose elements are pairs (g,ϑ) with multiplication law

$$(g,\vartheta)(g',\vartheta') = (gg',\vartheta+\vartheta'+\xi(g,g')) \tag{147}$$

For any local exponent, we have as a consequence of [143] or the requirement of associativity for this multiplication

$$\begin{cases}\xi(e,e) = \xi(g,e) = \xi(e,g') = 0 \\ \xi(g,g') + \xi(gg',g'') = \xi(g,g'g'') + \xi(g',g'')\end{cases} \tag{148}$$

Looking at the definition [141] we recognize that the second line in [148] is the statement that ξ is a 2-cocycle in $C_2(G,\mathbb{R})$. Accordingly, two local exponents are equivalent if a real valued continuous function $\zeta(g)$ exists such that

$$\xi' = \xi + \delta[\zeta] \tag{149}$$

where

$$\delta[\zeta](g,g') = \zeta(g) + \zeta(g') - \zeta(gg') \tag{150}$$

Taking [138] into consideration, it is apparent that our local exponent is

$$\begin{aligned}\xi\big((A,B),(A',B')\big) &= c\big[\chi^{t}(A,B) + \chi^{t}(A',B') - \chi^{t}\big((A,B)(A',B')\big)\big] = \\ &= \delta[c\,\chi^{t}]\big((A,B),(A',B')\big)\end{aligned} \tag{151}$$

and is therefore in the zero class of $H_2(G,\mathbb{R})$. The fact that we are dealing here with a connected, simply-connected Lie group G_c is enough to guarantee [Bar] the existence of an extension of the local exponent ξ to the entire group and the fact that the local exponent is equivalent to zero indicates that the induced unitary representation that ensues will be ordinary (i.e., one for which the factor of a unitary ray-representation is equivalent to 1). For the non-simply-connected cases, one has to resort to the universal covering group $\tilde{G}$ of G, which is simply connected, and in order to find the ray representations of G it is only necessary to select those of $\tilde{G}$ which map every element of a discrete central invariant subgroup $D \subset \tilde{G}$ into the unit ray. Here D is such that $G \equiv \tilde{G}/D$.

Given that the local properties of the neighborhood of the identity of a Lie group G are mainly determined by the structure of its Lie algebra $\mathcal{G}$, in most cases of interest the local exponents can be determined from the infinitesimal analysis on G. Following here the work of Bargman [Bar], define any real-valued antisymmetric bilinear form Ξ on the Lie algebra $\mathcal{G}$ which verifies the relation

$$[152] \qquad \delta[\Xi](\xi,\xi',\xi'') = \Xi([\xi,\xi'],\xi'') + \Xi([\xi',\xi''],\xi) + \Xi([\xi'',\xi],\xi') = 0$$

to be an *infinitesimal exponent*. Each local exponent ξ corresponds to one such infinitesimal exponent Ξ through the limit

$$[153] \qquad \Xi(\xi,\zeta) = \lim_{\tau\to 0} \tau^{-2}\{\xi(e^{-\tau\xi}, e^{-\tau\zeta}) + \xi(e^{\tau\xi}, e^{\tau\zeta}) + \xi(e^{\tau\xi}e^{\tau\zeta}, e^{-\tau\xi}e^{-\tau\zeta})\}$$

This expression actually follows from the computation of the commutation relations in the local group extension determined by ξ and the multiplication law [147], by performing the usual limit of the product $\tau^{-2}(e^{-\tau\xi},-\tau\lambda)(e^{-\tau\zeta},-\tau\kappa)(e^{\tau\xi},\tau\lambda)(e^{\tau\zeta},\tau\kappa)$ to define the Lie bracket of (ξ,λ) and (ζ,κ) in the extended algebra, which turns out to be

$$[154] \qquad [(\xi,\lambda),(\zeta,\kappa)] = ([\xi,\zeta],\ \Xi(\xi,\zeta))$$

To determine the possible infinitesimal exponents in practice, one has to evaluate [152] for a particular basis of $\mathcal{G}$, which is tantamount to solve the linear system for the components $\Xi_{\kappa\upsilon}$ of Ξ in that basis

$$[155] \qquad C^{\kappa}_{\sigma\rho}\Xi_{\kappa\upsilon} + C^{\kappa}_{\rho\upsilon}\Xi_{\kappa\sigma} + C^{\kappa}_{\upsilon\sigma}\Xi_{\kappa\rho} = 0$$

where $C^{\kappa}_{\alpha\beta}$ are the structure constants of $\mathcal{G}$. We have solved the corresponding problem for the dynamical group G_c and in the basis specified by $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ the possible infinitesimal exponents are of the form

$$[156] \qquad \Xi_{abc} = a\,\beta^1\wedge\beta^2 + b\,(\beta^1\wedge\beta^3 + \beta^2\wedge\beta^4) + c\,\beta^2\wedge\beta^3$$

with a, b and c arbitrary constants in $\mathbb{R}$ and β^{κ} the dual forms to the ξ_{κ}. The specific case of the exponent [151] corresponds to the infinitesimal exponent

$$[157] \qquad \Xi_0 = -\tfrac{1}{c}\,(\beta^1\wedge\beta^3 + \beta^2\wedge\beta^4)$$

Notice how this can be rewritten, when evaluated at arbitrary $\xi, \zeta \in \mathcal{G}$, as

$$[158] \qquad \Xi_0(\xi, \zeta) = \beta^4([\xi, \zeta]) = \delta[\beta^4](\xi, \zeta)$$

where the last identity is merely the restatement that the infinitesimal exponent Ξ_0 is associated with a local exponent ξ equivalent to zero.

In the following section, we will analize the representations allowed by the extended group W_E using the method of orbits and address the physical meaning of the quantization procedure associated with the geometric quantization programme.

CHAPTER 7
HARMONIC ANALYSIS ON W_ε AND GEOMETRIC QUANTIZATION

Given a Lie group G, the problem of obtaining the set $\hat{G}$ of its irreducible unitary representations (u.i.r.) has been completely solved by Kirillov [Kir] for the case of nilpotent groups. In his *method of orbits*, these representations may all be constructed from the orbits of the coadjoint representations of G on $\mathcal{G}^*$, the dual of its Lie algebra. Furthermore, to each such orbit corresponds a unique (equivalence class of) u.i.r.s. Viewed from another perspective, Kirillov's method of orbits yielded an inspiring example for a general program now known as *Geometric Quantization*, originally developed by Souriau [Sou] and Kostant [Kos], whereby symplectic mechanics, differential geometry and Lie group representation theory are brought together in an attempt to clarify the relations between the classical and quantum descriptions of Physics. In this connection, the nature of the coadjoint orbits of the dynamical group of a elementary system is critical in the geometric quantization process, since only those orbits $\mathcal{O}$ for which $\omega_{\mathcal{O}}$ determines an integer cohomology class in $H^2(\mathcal{O}, \mathbb{Z})$ will be *quantizable*, i.e., can possibly be associated with a unitary representation of G (except for nilpotent G, these will not be irreducible in general). But already at the classical level, this approach benefits from the fact that any G-homogeneous symplectic manifold is locally modeled by a coadjoint G-orbit, in the sense that one can always arrange its charts to be local coverings, *as symplectic manifolds*, of one such orbit (see below). Indeed, there is a stronger result that states that all G-homogeneous, hamiltonian symplectic manifold can be obtained as a covering of a coadjoint orbit of G.

In the case of type I groups, an arbitrary unitary representation can be expressed "uniquely" as a direct integral of irreducible representations. For connected simply connected *solvable* Lie groups, Auslander and Kostant [Aus] have shown that type I is equivalent to having every coadjoint orbit be the intersection of an open and a closed set, with a symplectic structure cohomologous to [0]. In this case unitary representations of G can again be obtained from the coadjoint orbits of G on $\mathcal{G}^*$ but, unlike the nilpotent case, several representations can be associated with the same orbit. Furthermore, they are not irreducible in general, although one can obtain irreducibles by introducing different polarizations.

There is, however, a special class of solvable groups for which the parametrization of $\hat{G}$ is given by the coadjoint orbits: this is the case of the *exponential* solvable Lie groups, i.e., those for which the map $\exp: \mathcal{G} \longrightarrow G$ is bijective. Then $\hat{G}$ is again in one-one correspondence with $\mathcal{G}^*/G$ and the irreducible representation associated with any orbit $\mathcal{O}$ is *induced* by a unique character χ and a *real* polarization $\mathfrak{h}$ suitably chosen.

We have shown in the end of the last chapter that the group G_c is indeed exponential, therefore type I. Here we analyze the orbit structure of the coadjoint action of the central extension W_c of G_c, and show that there is in this case a one parameter family of 4-dimensional orbits. From this we will conclude that, up to equivalence, there is a unique unitary irreducible representation of the C.C.R. determined by W_c once the parameter has been chosen to equal the experimental value of Planck's constant, just as in the flat case.

We recall that the *adjoint* action of a Lie group G on its Lie algebra $\mathcal{G}$ is

$$[159] \qquad \mathcal{A}d_g\,\xi = \frac{d}{dt}\Big|_{t=0}\left\{ g\exp(-t\xi)\, g^{-1} \right\}$$

where $g\in G$ and $\xi\in\mathcal{G}$. Consequently the *coadjoint* action of a Lie group G on the dual $\mathcal{G}^*$ of its Lie algebra $\mathcal{G}$ is defined as

$$[160] \qquad \mathcal{K}d_g\,\alpha = \alpha\cdot\mathcal{A}d_{g^{-1}} = \mathcal{A}d^*_{g^{-1}}\alpha \qquad\qquad (\forall\, \alpha\in\mathcal{G}^*,\, g\in G_c)$$

A notable characteristic common to any orbit $\mathcal{O}\subset\mathcal{G}^*$ of the coadjoint representation $\mathcal{K}d$ is that each possesses a canonical symplectic structure $\omega_{\mathcal{O}}$ [Kir] which is preserved by the coadjoint action, i.e., $\mathcal{K}d^*_g\omega_{\mathcal{O}} = \omega_{\mathcal{O}}$, thus these orbits are G-homogeneous symplectic manifolds. In fact, since $\mathcal{O}$ is generated by the action of G, the one parameter group through $\alpha \in \mathcal{O}$ determined by the flow of $\mathbf{X} \in T\mathcal{O}$ is of the form

[161] $$\exp(-t\mathbf{X})(\alpha) = \mathcal{A}d^{*}_{\exp(t\xi_{\mathbf{X}})}\,\alpha$$

where $\xi_{\mathbf{X}} \in \mathcal{G}$ is the Lie algebra element which generates a fundamental vector field $\mathbf{X}_{\xi}$ on $\mathcal{O}$ that takes the specified value $\mathbf{X}$ at α.

The canonical symplectic structure on $\mathcal{O}$ is then defined as

[162] $$\omega_{\mathcal{O}}(\mathbf{X},\mathbf{Y})\big|_{\alpha} = \iota_{[\xi_{\mathbf{X}},\xi_{\mathbf{Y}}]}\alpha = \alpha([\xi_{\mathbf{X}}, \xi_{\mathbf{Y}}]) \quad (\forall\ \alpha \in \mathcal{G}^{*},\ \mathbf{X}, \mathbf{Y} \in T\mathcal{O})$$

We will show now that any momentum mapping $\mu : M \longrightarrow \mathcal{G}^{*}$ for the symplectic action $\sigma : \mathcal{G} \longrightarrow \mathcal{C}an(M, \omega_M)$ on a G–homogeneous, hamiltonian symplectic manifold M is in fact a covering of some orbit $\mathcal{O} \subset \mathcal{G}^{*}$ of the action [160]. This means that the symplectic structure of M, ω_M, is the pull-back by μ of the canonical symplectic structure [162] on $\mathcal{O}$

[163] $$\omega_M = \mu^{*}\omega_{\mathcal{O}}$$

and that the hamiltonian structures of both manifolds are isomorphic, i.e.,

[164] $$\mu_{*}(\mathbf{X}^{M}_{\xi}) = \mathbf{X}^{\mathcal{O}}_{\xi}$$

Proof: Recall that the commutator between elements $\xi, \zeta \in \mathcal{G}$ can be viewed as the restriction to the identity of the tangent map to the *adjoint* action [159], and is frequently denoted as

[165] $$ad_{\xi}(\zeta) = \frac{d}{dt}\Big|_{t=0}\{\mathcal{A}d_{\exp(t\xi)}\zeta\} = [\,\xi, \zeta\,]$$

Note also that any element $\zeta \in \mathcal{G}$ can be viewed as a linear function $h_{\zeta}(\alpha) = \alpha(\zeta)$, for α in $\mathcal{O} \subset \mathcal{G}^{*}$, and thus it is legitimate to evaluate $\mathbf{X}^{\mathcal{O}}_{\xi}(h_{\zeta})$ at an arbitrary point $\alpha \in \mathcal{O}$

[166] $$\mathbf{X}^{O}_{\boldsymbol{\xi}}(f_{\boldsymbol{\zeta}})|_{\alpha} = \frac{d}{dt}\Big|_{t=0}\left\{f_{\boldsymbol{\zeta}}\left(\mathcal{K}d_{\exp(-t\boldsymbol{\xi})}(\alpha)\right)\right\} =$$

$$= \frac{d}{dt}\Big|_{t=0}\left\{\alpha\left(\mathcal{A}d_{\exp(t\boldsymbol{\xi})}\boldsymbol{\zeta}\right)\right\} = \alpha([\boldsymbol{\xi},\boldsymbol{\zeta}]) =$$

$$= \omega_{O}(\mathbf{X}^{O}_{\boldsymbol{\xi}},\mathbf{X}^{O}_{\boldsymbol{\zeta}})|_{\alpha}$$

If we now set α to be μ_m, we get from [166] and the definition of momentum map

[167] $$\mathbf{X}^{O}_{\boldsymbol{\xi}}(f_{\boldsymbol{\zeta}})|_{\mu_m} = \mu_m([\boldsymbol{\xi},\boldsymbol{\zeta}]) = f_{[\boldsymbol{\xi},\boldsymbol{\zeta}]}(m) = \{f_{\boldsymbol{\xi}}, f_{\boldsymbol{\zeta}}\}|_m = \mathbf{X}^{M}_{\boldsymbol{\xi}}(f_{\boldsymbol{\zeta}})|_m =$$

$$= \frac{d}{dt}\Big|_{t=0}\left\{f_{\boldsymbol{\zeta}}(\exp(-t\boldsymbol{\xi})\,m)\right\} =$$

$$= \frac{d}{dt}\Big|_{t=0}\left\{\mu_{\exp(-t\boldsymbol{\xi})m}(\boldsymbol{\zeta})\right\} = \mu_*(\mathbf{X}^{M}_{\boldsymbol{\xi}})|_{\mu_m}(\boldsymbol{\zeta}) =$$

$$= \mu_*(\mathbf{X}^{M}_{\boldsymbol{\xi}})(f_{\boldsymbol{\zeta}})|_{\mu_m}$$

This therefore proves [164], whereas [163] follows as a consequence of [162] and

[168] $$\omega_M(\mathbf{X}^{M}_{\boldsymbol{\xi}},\mathbf{X}^{M}_{\boldsymbol{\zeta}})|_m = f_{[\boldsymbol{\xi},\boldsymbol{\zeta}]}(m) = \mu_m([\boldsymbol{\xi},\boldsymbol{\zeta}]) = \omega_O(\mathbf{X}^{O}_{\boldsymbol{\xi}},\mathbf{X}^{O}_{\boldsymbol{\zeta}})|_{\mu_m} =$$

$$= \omega_O(\mu_*(\mathbf{X}^{M}_{\boldsymbol{\xi}}),\mu_*(\mathbf{X}^{M}_{\boldsymbol{\zeta}}))|_{\mu_m} =$$

$$= \mu^*\omega_O(\mathbf{X}^{M}_{\boldsymbol{\xi}},\mathbf{X}^{M}_{\boldsymbol{\zeta}})|_m$$

The pratical way of computing the coadjoint action is to relate it to the adjoint action of the group on its Lie algebra, using the definition [159] above. First, we agree on using the more convenient coordinate system on W_c, i.e., the one in which the multiplication law between $\{\theta, g\}$ and $\{\theta', g'\}$ in W_c is expressed as

[169] $$\{\theta, g\}\{\theta', g'\} = \{\theta, (\mathbf{A},\mathbf{B})\}\{\theta', (\mathbf{A}',\mathbf{B}')\} =$$

$$= \{\theta + \theta' + \delta[c\,\chi^{k}]((\mathbf{A},\mathbf{B}),(\mathbf{A}',\mathbf{B}')),\ (\mathbf{A},\mathbf{B})(\mathbf{A}',\mathbf{B}')\}$$

where

$$[170]\qquad \begin{cases} \chi^{\delta}(\mathbf{A},\mathbf{B}) = B_2\, h(A^2) - A^1 B_1\, h'(A^2) \\ (\mathbf{A},\mathbf{B})(\mathbf{A}',\mathbf{B}') = \left(\mathbf{A} + \left(1-\frac{A^2}{c}\right)\mathbf{A}',\ \mathbf{B} + \left(1-\frac{A^2}{c}\right)\mathbf{B}' - \frac{B_1}{c} J\, \mathbf{A}'\right) \\ \delta[c\,\chi^{\delta}]\left((\mathbf{A},\mathbf{B}),(\mathbf{A}',\mathbf{B}')\right) = c\left[\chi^{\delta}(\mathbf{A},\mathbf{B}) + \chi^{\delta}(\mathbf{A}',\mathbf{B}') - \chi^{\delta}\left((\mathbf{A},\mathbf{B})(\mathbf{A}',\mathbf{B}')\right)\right] \end{cases}$$

and $h(Q^2) = \frac{c}{Q^2}\, ln\left\{1-\frac{Q^2}{c}\right\}$, with $h' = \frac{dh}{dQ^2}$, as defined previously

Accordingly, a convenient basis for the Lie algebra $\mathcal{W}_c^1$ is such that a generic element in it can be represented as

$$[171]\qquad \xi_{(\rho,\mathbf{a},\mathbf{b})} = \rho\,\xi_0 + \sum_{e=1}^{2}\left[a^e \xi_e + b^e \xi_{e+2}\right] \in \mathcal{W}_c$$

in which case the one-parameter subgroup that $\xi_{(\rho,\mathbf{a},\mathbf{b})}$ generates is described by

$$[172]\qquad \exp\left(-t\,\xi_{(\rho,\mathbf{a},\mathbf{b})}\right) = \left(t\rho,\ (\mathbf{A}(t,\mathbf{a},\mathbf{b}),\ \mathbf{B}(t,\mathbf{a},\mathbf{b}))\right)$$

where

$$[173]\qquad \begin{cases} \mathbf{A}(t,\mathbf{a},\mathbf{b}) = \frac{c}{a^2}\left(1-e^{-\frac{a^2 t}{c}}\right)\mathbf{a} \\ \mathbf{B}(t,\mathbf{a},\mathbf{b}) = \frac{c}{a^2}\left(1-e^{-\frac{a^2 t}{c}}\right)\mathbf{b} + \frac{b_1}{a^2}\left\{t\, e^{-\frac{a^2 t}{c}} - \frac{c}{a^2}\left(1-e^{-\frac{a^2 t}{c}}\right)\right\} J\,\mathbf{a} \end{cases}$$

Using these definitions in [159], we are thus able to calculate, for $\{\theta, g\} = \{\theta, (\mathbf{A},\mathbf{B})\}$

$$[174]\qquad \mathcal{Ad}_{(\theta,g)}\xi_{(\rho,\mathbf{a},\mathbf{b})} = \xi_{(\rho(\theta,g),\mathbf{a}(\theta,g),\mathbf{b}(\theta,g))}$$

with

$$[175]\qquad \begin{cases} \rho(\theta,g) = \rho + \mathbf{b}\cdot\mathbf{A} - \mathbf{a}\cdot\mathbf{B} \\ \mathbf{a}(\theta,g) = \left(1-\frac{A^2}{c}\right)\mathbf{a} + \frac{a^2}{c}\mathbf{A} \\ \mathbf{b}(\theta,g) = \left(1-\frac{A^2}{c}\right)\mathbf{b} - \frac{B^1}{c} J\,\mathbf{a} + \frac{a^2}{c}\mathbf{B} + \frac{b^1}{c} J\,\mathbf{A} \end{cases}$$

Evaluating now the result [174] with the linear form $\Omega = \{\gamma, (\boldsymbol{\alpha}, \boldsymbol{\beta})\} \in \mathcal{W}_c^{*}$ we obtain the desired expression for the coadjoint action, albeit in terms of $\{\theta, g\}^{-1}$

[176] $$\mathcal{K}d_{(\theta,g)^{-1}}\Omega = \mathcal{A}d^*_{(\theta,g)}\Omega = \Omega\circ\mathcal{A}d_{(\theta,g)} = (\gamma(\theta,g), (\boldsymbol{\alpha}(\theta,g), \boldsymbol{\beta}(\theta,g)))$$

where

[177] $$\begin{cases} \gamma(\theta,g) = \gamma \\ \boldsymbol{\alpha}(\theta,g) = \boldsymbol{\alpha} + (\frac{\beta_2}{c}+\gamma)\mathbf{B} + \frac{\alpha_1}{c}\mathbf{J}\,\mathbf{A} \\ \boldsymbol{\beta}(\theta,g) = \boldsymbol{\beta} - (\frac{\beta_2}{c}+\gamma)\mathbf{A} \end{cases}$$

We immediately conclude from this that γ is one of the parameters that distinguishes different orbits, since it is not affected by the coadjoint action. Next, we observe that the conditions

[178] $$\begin{cases} \alpha_1 = 0 \\ \beta_2 = -c\gamma \end{cases}$$

determine all 0-dimensional orbits, i.e., all the fixed points for the coadjoint action of W_c on $\mathcal{W}_c^*$. Then again, the conditions

[179] $$\begin{cases} \alpha_1 \neq 0 \\ \beta_2 = -c\gamma \end{cases}$$

determine all 2 dimensional orbits, and there is a 2–parameter family of these, with parameters $(\gamma, \beta_1) \in \mathbb{R}^2$. Finally, the condition

[180] $$\beta_2 \neq -c\gamma$$

is necessary to determine all the 4-dimensional orbits, and in fact there is a doubly–degenerate, one-parameter family of such, indexed by the value γ. We recall here that, whereas $\mathbf{B} \in \mathbb{R}^2$, $\mathbf{A}$ is constrained by the requirement $-\infty < \mathbf{A}^2 < c$. This means that these 4-dimensional orbits have a boundary at finite distance, determined by the equation

[181] $$\beta_2(\theta,g) = \beta_2 - (\tfrac{\beta_2}{c}+\gamma)\,\mathbf{A}^2$$

When $\beta_2 < -c\gamma$, equation [181] implies that

$$[182] \qquad -\infty < \beta_2(\theta, g) < -c\gamma$$

and when $-c\gamma < \beta_2$, we verify that

$$[183] \qquad -c\gamma < \beta_2(\theta, g) < +\infty$$

Furthermore, given β_2 and $\tilde{\beta}_2$ in one of the domains [182] or [183], it is easy to show that there is a unique value of A^2, $(-\infty < A^2 < c)$, such that

$$[184] \qquad \tilde{\beta}_2 = \beta_2 - \left(\frac{\beta_2}{c} + \gamma\right) A^2$$

This therefore proves that, apart from the value γ, only the direction of the inequality [180] is in addition necessary to uniquely determine one of the 4-dimensional coadjoint orbits. We have thus obtained a classification of all the coadjoint orbits of our solvable group $\mathbf{W}_c$, and one of these must be, as we have previously shown, *symplectomorphic* to our phase-space. Recalling that the symplectic form in phase space T^*Q_c is given by [89] in *right* hamiltonian coordinates [87], we see that the hamiltonian vector field associated with $f \in \mathcal{C}^\infty_{\mathbb{R}}(T^*Q_c)$ is

$$[185] \qquad \mathbf{X}_f = \left(1 - \frac{Q^2}{c}\right)(\partial_\mathbf{P} f)\, \partial_\mathbf{Q} - \left\{\left(1 - \frac{Q^2}{c}\right)(\partial_\mathbf{Q} f) + \frac{P_1}{c} J(\partial_\mathbf{P} f)\right\} \partial_\mathbf{P}$$

and an elementary calculation reveals that, for

$$[186] \qquad \xi_{(\rho,\mathbf{a},\mathbf{b})} = \rho\xi_0 + \sum_{e=1}^{2}\{a^e\xi_e + b^e\xi_{e+2}\} \in \mathcal{W}_c,$$

we can define a co-momentum map $\hat{\mu}: \mathcal{W}_c \longrightarrow \mathcal{C}^\infty_{\mathbb{R}}(T^*Q_c)$ whose action is of the form

$$[187] \qquad \hat{\mu}(\xi_{(\rho,\mathbf{a},\mathbf{b})})(\mathbf{Q}, \mathbf{P}) = \mathbf{a} \cdot \mathbf{P} + \mathbf{b} \cdot \mathbf{Q} + \rho \in \mathcal{C}^\infty_{\mathbb{R}}(T^*Q_c)$$

In this case, the momentum map is simply

$$[188] \qquad \mu_{(\mathbf{Q},\mathbf{P})} = \beta^0 + \sum_{e=1}^{2}\{P_e\beta^e + Q^e\beta^{e+2}\} \in \mathcal{W}_c^*$$

where the β^i are the dual forms to the ξ_i. The map [188] is, as expected, a covering from the symplectic manifold T^*Q_ϵ to the orbit determined by $\gamma = 1, \beta_2 < \epsilon$

For the purposes of quantum mechanics, we are interested in finding all the possible unitary, irreducible representations of W_ϵ. The generalization of Kirillov's results to solvable groups has been done by Auslander and Kostant [Aus] who showed that, for G a connected, simply-connected solvable Lie group, there exists a strongly admissible positive polarization $\mathfrak{h}$ at any point $\alpha \in \mathcal{G}^*$ and that any such polarization satisfies the Pukansky condition (which for exponential groups reads $\mathcal{Ad}_H \alpha = \alpha + \mathfrak{h}^\perp$, where $\mathfrak{h}^\perp \subseteq \mathcal{G}^*$ denotes the space of linear functionals vanishing on $\mathfrak{h}$ and $H = G_\alpha H_0$, here $H_0 \subset G$ is the connected component of the subgroup of G with Lie algebra $\mathfrak{h}$ and $G_\alpha \subset G$ is the stabilizer of α). Then, when α belongs to an integral coadjoint orbit O (i.e., the natural symplectic structure [162] defines an integer cohomology class in $H^2(O,\mathbb{Z})$) a character η_α can be defined on G_α whose differential is the restriction of $2\pi i\alpha$ to the Lie algebra $\mathcal{G}_\alpha$ of G_α; from this character η_α and polarization $\mathfrak{h}$ at α, the induced unitary representation $\mathrm{ind}_G(\eta_\alpha, \mathfrak{h})$ of G, may be formed which is irreducible and independent of the positive, strongly admissible polarization $\mathfrak{h}$ at α. Furthermore, if G is of Type I any α is in an integral co-orbit, and every irreducible unitary representation of G is of the form $\mathrm{ind}_G(\eta_\alpha, \mathfrak{h})$ (a connected, simply-connected Lie group is of Type I if and only if the symplectic structure of every coadjoint orbit O is in the zero class of $H^2(O,\mathbb{R})$, and every O is the intersection of a closed and an open set in $\mathcal{G}^*$; these conditions are realized for our group W_ϵ). Finally, two representations of this form are equivalent if and only if they are induced from characters η_{α_1} and η_{α_2} at points α_1 and α_2 in the same co-orbit, such that $L_g^* \eta_{\alpha_2} = \eta_{\alpha_1}$, where $g \in G$ and $L_g G_{\alpha_1} = G_{\alpha_2}$

The most notable difference between Kirillov's results for nilpotent groups and the representation theory for solvable Type I groups resides on the fact that, in general, the bijective correspondence between coadjoint orbits $\mathcal{G}^*/G$ and unitary irreducible representations of G no longer holds since the character η_α is not necessarily unique

This is because the stabilizer G_{α} can have non trivial topology, and if we denote by $\Pi_1^c(\mathcal{O})$ the group of characters for G_{α}/G_{α}^0 (the quotient of the stabilizer of a point $\alpha \in \mathcal{O}$ by its connected component of the identity, G_{α}^0) – which, is naturally isomorphic to the fundamental group $\Pi_1(\mathcal{O})$, hence the notation – we can obtain all the characters at α by multiplying one of them by the elements of $\Pi_1^c(\mathcal{O})$. A general result in geometric quantization [Kos] equates this ambiguity with the set of all equivalence classes of line-bundles with-connection $\mathcal{L}(\mathcal{O}\,\nabla\,h)$ on $\mathcal{O}$, whose curvature is $2\pi i\,\omega_{\mathcal{O}}$ and which possess a compatible hermitean structure h. Thus the generalization of Kirillov's theorem to a connected, simply-connected solvable Lie group G of Type I can only state that the u.i.r.'s of G are parametrized by the union, over all coadjoint orbits $\mathcal{O}$, of the set of equivalence classes of hermitean line-bundles with-connection over $\mathcal{O}$. Only when *every* orbit $\mathcal{O} \in \overset{*}{\mathcal{G}}/G$ is simply-connected can we assert the bijection between co-orbits and u.i.r.'s of G, and this happens always for the exponential case. Since we already saw that our group W_c is exponential, the problem of finding all u.i.r. is completely solved; thus, we can assert that the von Neumann-Stone-Mackey uniqueness theorem extends from the usual case where the configuration space ($\mathbb{R}^n$) is flat, to the curved situation (Q_c) considered in this dissertation.

CONCLUSION

The results obtained in the preceding chapters prove that it is possible to perform a mathematically sound quantization in spaces of constant negative curvature, and they show why the necessary Canonical Commutation Relations *cannot* be postulated to be the ones attached to the traditional Heisenberg algebra. when applied to situations other than that of a flat configuration space, these can merely hope to describe a linearized version of the dynamics in a local region As was already noted by Isham [Ish], a general quantization procedure on an arbitrary configuration space Q must include the determination of an appropriate analogue of the flat C.C.R., and it was our purpose here to show how this can actually be implemented in a specific example.

The choice of the 2-dimensional Poincaré half plane Q_c was motivated by a previous work of Prof Emch [Emc] where starting from the quantum mechanics as formalized by Mackey's [Mac] systems of imprimitivity for homogeneous Riemannian manifolds, one obtains the Jordan and Lie structures of the classical theory from purely quantum principles via a classical limit $\hbar \to 0$ of the initial Mackey's axiomatization It was shown there that the classical limit implements a correspondence principle from quantum to classical observables, that in the curved case his process of *geometric dequantization* determines unambiguously the classical momenta observables associated to the configuration space coordinates, and that these differ from the Darboux coordinates usually taken as the canonical momenta. My approach here, being the reverse path of that paper, allows us to confirm and better understand the nature of those momenta observables they are precisely the hamiltonian functions of the vector fields that generate the subgroup H_c of the full isometry group of the configuration space Q_C, i e., the generators of translations in configuration space.

The thrust of this dissertation concentrated on finding the correct canonical group acting transitively on the phase space T^*Q_c by using a differential geometric approach, and then on implementing the geometric quantization methodology to classify the possible quantizations of the new C.C.R. We were thus able to show that there is a generalization of the Stone von Neumann theorem to the curved configuration manifold Q_c, analogous to the one known for $Q_0 \equiv \mathbb{R}^n$ in that there is a one parameter family (parametrized by the value of $\hbar$) of inequivalent classes of irreducible unitary representations of the Weyl-Heisenberg group W_c, and each of these classes is equivalent to an irreducible representation obtained from a co-adjoint orbit $\mathcal{O} \subset \mathcal{W}_c^*$ by induction of a character on the orbit, chosen together with a strongly admissible real polarization, that we know always exist since we show that W_c is a solvable, simply connected, exponential Lie group

APPENDIX

Let M represent a n-dimensional manifold We assume here that the domains $\mathcal{U}_\alpha$ and local trivializations $\varphi_\alpha\ \mathcal{U}_\alpha \subset M \to \mathbb{R}^n$ are known for a given atlas of M We will also assume throughout this work, that the summation convention on repeated indices $(i,j \quad = 1,2, \quad, n)$ is enforced and denote for simplicity

[n 1] $\mathbf{q} = q^i\,\mathbf{e}_i$ (dropping the subscript α as long as a particular choice of coordinate patch is consistently chosen) the coordinate map giving the local representation of any point in M as an element in $\mathbb{R}^n$ When no confusion arises, we will refer freely to a point $m \in M$ as the *point* $\mathbf{q} \in \mathbb{R}^n$, it being understood that we mean the point whose local coordinates in a particular chart $\mathcal{U}_\alpha$ is given by $\varphi_\alpha(m) = \mathbf{q}\,(m) = q^i(m)\,\mathbf{e}_i$

[n-2] $(\mathbf{q}, \mathbf{p}) = (q^i\,\mathbf{e}_i\,, p_j\,\mathbf{e}^j\,)$ the coordinate map giving the local representation of any point in T^*M and its local representation as an element of $\mathbb{R}^n \times (\mathbb{R}^n)^*$

[n 3] $\partial_{\mathbf{q}} = \mathbf{e}^i \otimes \partial_{q^i}$ $\left(\text{resp } \partial_{\mathbf{p}} = \mathbf{e}_j \otimes \partial_{p_j}\right)$ a $(\mathbb{R}^n)^*$-valued (resp $\mathbb{R}^n$-valued) differential operator, acting on $C^\infty(T^*M)$ by the assignments $\partial_{\mathbf{q}} f = (\partial_{q^i} f)\mathbf{e}^i$, $\left(\text{resp } \partial_{\mathbf{p}} f = (\partial_{p_j} f)\mathbf{e}_j\right)$

[n-4] $d\mathbf{q} = dq^i \otimes \mathbf{e}_i$ $\left(\text{resp. } d\mathbf{p} = dp_j \otimes \mathbf{e}^j\right)$ a $\mathbb{R}^n$-valued (resp $(\mathbb{R}^n)^*$ valued) 1 form on T^*M, which we can 'contract' in two ways

[n 5] $d\mathbf{q}\,\partial_{\mathbf{q}} = dq^i \otimes \partial_{q^i}$ $\left(\text{resp } d\mathbf{p}\,\partial_{\mathbf{p}} = dp_j \otimes \partial_{p_j}\right)$ i e, the contraction with respect to duality in $\mathbb{R}^n$,

[n 6] $\iota_{\partial_{\mathbf{q}}} d\mathbf{q} = \mathbf{e}^i \otimes \mathbf{e}_i = 1$ $\left(\text{resp } \iota_{\partial_{\mathbf{p}}} d\mathbf{p} = \mathbf{e}_j \otimes \mathbf{e}^j = 1^*\right)$ i e, the contraction with respect to duality in $T(T^*M)$

These conventions are of use in writing index free local expressions for objects in [n-7] $TM \ni \mathbf{X} = \mathbf{v}\,\partial_{\mathbf{q}}$ with $\mathbf{v} = v^i \mathbf{e}_i \in \mathcal{F}(M, \mathbb{R}^n)$;

[n-8] $T^*M \ni \alpha = \mathbf{dq}\,\mathbf{n}$ with $\mathbf{n} = n_i \mathbf{e}^i \in \mathcal{F}(M, \mathbb{R}^{n*})$;

[n-9] $T(T^*M) \ni \check{\mathbf{X}} = \mathbf{u}\,\partial_{\mathbf{q}} + \mathbf{w}\,\partial_{\mathbf{p}}$ with $\mathbf{u} \in \mathcal{F}(T^*M, \mathbb{R}^n)$ and $\mathbf{w} \in \mathcal{F}(T^*M, \mathbb{R}^{n*})$;

[n-10] $T^*(T^*M) \ni \check{\alpha} = \mathbf{dq}\,\mathbf{a} + \mathbf{dp}\,\mathbf{b}$ with $\mathbf{a} \in \mathcal{F}(T^*M, \mathbb{R}^{n*})$ and $\mathbf{b} \in \mathcal{F}(T^*M, \mathbb{R}^n)$.

For example, the derivation map on $C^\infty(T^*M)$ can be written as

[n-11] $\mathbf{d}f(\mathbf{p},\mathbf{q}) = \left(\mathbf{dq}\,\partial_{\mathbf{q}} + \mathbf{dp}\,\partial_{\mathbf{p}}\right) f(\mathbf{p},\mathbf{q})$

We also define the exterior products

[n-12] $\mathbf{dq} \wedge \mathbf{dq} = dq^i \wedge dq^j \otimes \mathbf{e}_i \otimes \mathbf{e}_j$ $\left(\text{resp. } \mathbf{dp} \wedge \mathbf{dp} = dp_i \wedge dp_j \otimes \mathbf{e}^i \otimes \mathbf{e}^j\right)$

[n-13] $\mathbf{dp} \wedge \mathbf{dq} = dp_j \wedge dq^i \otimes \mathbf{e}^j \otimes \mathbf{e}_i$

With this definition, we note that

[n-14] $\mathbf{dq} \wedge \mathbf{dq} = -\,\mathbf{dq} \wedge \mathbf{dq}\,\check{\tau}$ where $\check{\tau} = \mathbf{e}^j \otimes \mathbf{e}^i \otimes \mathbf{e}_i \otimes \mathbf{e}_j$

[n-15] $\mathbf{dp} \wedge \mathbf{dp} = -\,\mathbf{dp} \wedge \mathbf{dp}\,\hat{\tau}$ where $\hat{\tau} = \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}^j \otimes \mathbf{e}^i$

and likewise

[n-16] $\mathbf{dp} \wedge \mathbf{dq} = -\,\mathbf{dq} \wedge \mathbf{dp}\,\tau$ where $\tau = \mathbf{e}_i \otimes \mathbf{e}^j \otimes \mathbf{e}^i \otimes \mathbf{e}_j$

Finally, we define the exterior derivative of a 1-form $\check{\alpha} = \mathbf{dq}\,\mathbf{a}(\mathbf{p},\mathbf{q}) + \mathbf{dp}\,\mathbf{b}(\mathbf{p},\mathbf{q}) \in T^*(T^*M)$ as

[n-17] $\mathbf{d}\check{\alpha} = -\,\mathbf{dq} \wedge \mathbf{da} - \mathbf{dp} \wedge \mathbf{db} =$

$$= -\,\mathbf{dq} \wedge \mathbf{dq}\left(\partial_{\mathbf{q}}\mathbf{a}\right) - \mathbf{dq} \wedge \mathbf{dp}\left(\partial_{\mathbf{p}}\mathbf{a} - \tau\,\partial_{\mathbf{q}}\mathbf{b}\right) - \mathbf{dp} \wedge \mathbf{dp}\left(\partial_{\mathbf{p}}\mathbf{b}\right) =$$

$$= -\,dq^i \wedge dq^j \otimes \mathbf{e}_i \otimes \mathbf{e}_j \left(\mathbf{e}^i \otimes \mathbf{e}^j\right) \partial_{q^i} a_j -$$

$$-\,dq^i \wedge dp_j \otimes \mathbf{e}_i \otimes \mathbf{e}^j \left(\mathbf{e}_j \otimes \mathbf{e}^i\right)\left(\partial_{p_j} a_i - \partial_{q^i} b^j\right) -$$

$$-\,dp_i \wedge dp_j \otimes \mathbf{e}^i \otimes \mathbf{e}^j \left(\mathbf{e}_i \otimes \mathbf{e}_j\right) \partial_{p_i} b^j =$$

$$= \left(\partial_{q^i} a_j\right) dq^i \wedge dq^j + \left(\partial_{p_j} a_i - \partial_{q^i} b^j\right) dp_j \wedge dq^i + \partial_{p_i} b^j\, dp_i \wedge dp_j$$

Note In some rare instances, it might be necessary to 'raise' or 'lower' indices for some of these objects With that in mind, we define

[n 18] $\delta^{\bullet} = \delta_{ij}\, e^{i} \otimes e^{j}$ (resp $\delta = \delta^{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j$)

and

[n-19] $\mathbf{dp} = \mathbf{dp}\,\delta = \delta^{ij}\, dp_i \otimes \mathbf{e}_j$

[n-20] $\partial^{\bullet}_{\mathbf{p}} = \delta^{\bullet}\,\partial_{\mathbf{p}} = \delta_{ij}\, e^{i} \otimes \partial_{p_j}$

If N is another k-dimensional differentiable manifold, and $\rho : M \longrightarrow N$ is a differentiable mapping we recall that the *pull-back* $\rho^{*}f$ to M, of a function $f \in C^{\infty}(N)$, is defined by

[n 21] $\rho^{*}f = f \circ \rho \in C^{\infty}(M)$

If we denote by $\mathbf{Q} = Q^{i}\mathbf{e}_i$ a (local) coordinate system on a chart of N, we give the *derivative map* $D\rho$ of ρ the convenient local representation

[n 22] $D\rho = \partial_{\mathbf{q}}(\rho^{*}\mathbf{Q})$

and this in turn justifies the local representation of the *tangent map* $T\rho$ to ρ as

[n 23] $T\rho = \mathbf{dq}\, \partial_{\mathbf{q}}(\rho^{*}\mathbf{Q})\, \partial_{\mathbf{Q}}$

It is now straightforward to see that, given a vector field $\mathbf{X} = \mathbf{v}\,\partial_{\mathbf{q}} \in TM$ we can define its *push forward* by ρ as the vector field $\rho_{*}\mathbf{X} \in TN$, given by its definition on the image of M by ρ

[n-24] $\rho_{*}\mathbf{X}\big|_{\rho(m)} = \left(\iota_{\mathbf{X}}\, T\rho\right)\big|_{m} = \left(\mathbf{v}\,\partial_{\mathbf{q}}(\rho^{*}\mathbf{Q})\right)\big|_{m}\,\partial_{\mathbf{Q}}$

Likewise, we will define the *pull-back* by ρ, of a 1 form $\boldsymbol{\beta} = \mathbf{dQ}\cdot\mathbf{b}$ defined on N, to a 1-form $\rho^{*}\boldsymbol{\beta}$ on M, by

[n 25] $\rho^{*}\boldsymbol{\beta}\big|_{m} = \iota_{T\rho\,|_{m}}\left(\boldsymbol{\beta}\big|_{\rho(m)}\right) = \mathbf{d}\left(\partial_{\mathbf{q}}(\rho^{*}\mathbf{Q})\,\rho^{*}\mathbf{b}\right)\big|_{m}$

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BIOGRAPHICAL SKETCH

AMARO JOSÉ RICA DA SILVA, Portuguese, was born in Lourenço Marques, Mozambique, in 1955, and moved to Lisbon when he was five. He studied Architecture at the Escola Superior de Belas Artes de Lisboa and then transferred to Physics at the Faculdade de Ciencias de Lisboa, where he obtained his Licenciatura in 1980. Since October that year he developed research at the Instituto de Física Matemática in Lisbon and is a teaching member of the Physics Department of the Instituto Superior Técnico de Lisboa, from which he is on a leave of absence. In 1984 he was selected to participate in the Fulbright Commission's Exchange Visitor Program and was offered a Teaching Fellowship in the Physics Department of the University of Rochester, where he obtained, in May of 1986, the degree of Master of Arts in Physics. From 1984 to 1988 he was the recipient of a INVOTAN-JNICT scholarship to conduct his predoctoral research. Since August, 1986, he has been a Teaching Assistant with the Physics Department of the University of Florida.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy

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This dissertation was submitted to the Graduate Faculty of the Department of Physics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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